THE OLYMPIAD CORNER

No. 318

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à crux-olympiad@cms.math.ca *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er avril 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier d'avoir traduit les problèmes.

OC156. Soit $ABCD$ un tétraèdre. Démontrer que le sommet D, le centre de la sphère inscrite et le centroïde de $ABCD$ sont colinéaires si et seulement si les surfaces des triangles ABD, BCD et CAD sont égales.

OC157. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$f(f(x)^2 + f(y)) = xf(x) + y, \ \forall x, y \in \mathbb{R}\,.$$

OC158. Démonter qu'un graphe fini, simple et planaire possède une orientation telle que tout sommet ait un degré extérieur au plus égal à 3.

OC159. Soit p un nombre premier impair. Démontrer qu'il existe un nombre naturel x tel que x et $4x$ sont toutes deux des racines primitives modulo p.

OC160. Le cercle inscrit du triangle ABC est tangent aux côtés BC, CA et AB à D, E et F respectivement. Soit T la réflexion de F par rapport à B et soit S la réflexion de E par rapport à C. Démontrer que le centre du cercle inscrit du triangle AST est à l'intérieur ou sur le cercle inscrit du triangle ABC.

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OC156. Let $ABCD$ be a tetrahedron. Prove that the vertex D, the center of the insphere and the centroid of $ABCD$ are collinear if and only if the areas of triangles ABD, BCD and CAD are equal.

OC157. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + f(y)) = xf(x) + y,\ \forall x, y \in \mathbb{R}\,.$$

OC158. Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.

OC159. Let p be an odd prime number. Prove that there exists a natural number x such that x and $4x$ are both primitive roots modulo p.

OC160. The incircle of triangle ABC is tangent to sides BC, CA and AB at D, E respectively F. Let T be the reflection of F with respect to B and S the reflection of E with respect to C. Prove that the incenter of triangle AST is inside or on the incircle of triangle ABC.

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OLYMPIAD SOLUTIONS

OC96. Let $a, b > 1$ be two relatively prime integers. We define $x_1 = a, x_2 = b$ and

$$x_n = \frac{{x_{n-1}}^2 + {x_{n-2}}^2}{x_{n-1} + x_{n-2}} \quad \forall n \geq 3\,.$$

Prove that x_n is not an integer for all $n \geq 3$.

Originally question 4 from Croatia, Team Selection Test 2011, Day 1.

There were no solutions received to this problem.

OC97. Let A be a set with 225 elements. Suppose that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \ldots \cup A_{11}| \geq 165$, and give an example for which equality holds.

Originally question 6 from 2011 USA Math Olympiad.

Solved by Oliver Geupel and we present his solution.

Without loss of generality we can assume that $A_1 \cup A_2 \cup \ldots \cup A_{11} = \{1, 2, \ldots, n\}$, where n is the cardinality of $A_1 \cup A_2 \cup \ldots \cup A_{11}$. We have to prove that $n \geq 165$.

For each $1 \leq i \leq n$ and $1 \leq j \leq 11$, we define

$$e_{ij} = \begin{cases} 0 & \text{if } i \notin A_j \\ 1 & \text{if } i \in A_j\,. \end{cases}$$

Then,

$$495 = 11 \cdot 45 = \sum_{j=1}^{11} |A_j| = \sum_{i=1}^{n} \sum_{j=1}^{11} e_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{11} e_{ij}^2\,.$$

By the Arithmetic-Quadratic Means inequality, we have

$$\begin{aligned} \left(\sum_{i=1}^{n} \sum_{j=1}^{11} e_{ij} \right)^2 \leq n \sum_{i=1}^{n} \left(\sum_{j=1}^{11} e_{ij} \right)^2 &= n \left(\sum_{j=1}^{11} \sum_{i=1}^{n} e_{ij}^2 + 2 \sum_{1 \leq j < k \leq 11} \sum_{i=1}^{n} e_{ij} e_{ik} \right) \\ &= n \left(495 + 2 \sum_{1 \leq j < k \leq 11} |A_j \cap A_k| \right) \\ &= n \cdot \left(495 + 2 \binom{11}{2} \cdot 9 \right) = 1485n\,. \end{aligned}$$

We conclude $n \geq \dfrac{495^2}{1485} = 165$, which completes the proof.

It remains to give an example for which $n = 165$ holds. The set $\{1, 2, \ldots, 11\}$ has $\binom{11}{3} = 165$ subsets of cardinality 3. Let $B_1,\ B_2,\ \ldots,\ B_{165}$ be any list of these subsets. Define $A_1, \ldots, A_{11}$ by the condition

$$i \in A_j \Leftrightarrow j \in B_i, \qquad 1 \le i \le 165, \quad 1 \le j \le 11\,.$$

Every number $j \in \{1, 2, \ldots, 11\}$ is included in exactly $\binom{10}{2} = 45$ of the subsets B_i. Thus, $|A_j| = 45$. Moreover, for any fixed distinct values $j, k \in \{1, 2, \ldots, 11\}$, we have $i \in A_j \cap A_k$ if and only if $j, k \in B_i$, which is satisfied for nine subsets B_i. Therefore, $|A_j \cap A_k| = 9$. This completes our example.

OC98. Let ABC be a triangle with $\angle BAC = 60^\circ$. Let B_1 and C_1 be the feet of the bisectors from B and C. Let A_1 be the symmetrical of A with respect to the line B_1C_1. Prove that A_1, B and C are colinear.

Originally question 3 from Moldova's Team Selection Test 2011, Day 2.

Solved by Š. Arslanagić; M. Bataille; and T. Zvonaru and N. Stanciu. We give the solution of Šefket Arslanagić.

Let I denote the intersection of BB_1 and CC_1, that is I is the incenter of ABC. Let $\angle B = \beta, \angle C = \gamma$. We know that $\beta + \gamma = 120^\circ$. Then, we have

$$\angle B_1IC_1 = \angle BIC = 180^\circ - \frac{\beta}{2} - \frac{\gamma}{2} = 180^\circ - 60^\circ = 180^\circ - \angle A\,.$$

This implies that the quadrilateral AB_1IC_1 is cyclic, and hence

$$\angle AB_1C_1 = \angle AIC_1\,.$$

As AIC_1 is an exterior angle for the triangle AIC, it follows that

$$\angle AB_1C_1 = \angle AIC_1 = \angle IAC + \angle ICA = 30^\circ + \frac{\gamma}{2}\,.$$

As A_1 is the symmetric of A with respect to B_1C_1, we have $\Delta AB_1C_1 \equiv \Delta A_1B_1C_1$ and hence

$$\angle A_1B_1C_1 = \angle AB_1C_1 = 30^\circ + \frac{\gamma}{2},$$
$$\angle B_1A_1C_1 = \angle B_1AC = 60^\circ.$$

Since AB_1IC_1 is cyclic, we also get

$$\angle IB_1C_1 = \angle IAC_1 = 30^\circ,$$
$$\angle IC_1B_1 = \angle IAB_1 = 30^\circ.$$

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As $\angle CB_1A_1 + \angle A_1B_1A = 180^\circ$, it follows that

$$\angle C_1B_1A = 180^\circ - \angle AB_1C_1 - \angle A_1B_1C_1 = 180^\circ - 2(30^\circ + \frac{\gamma}{2}) = \beta\,.$$

By the bisector theorem we have

$$\frac{BA}{BC} = \frac{AB_1}{B_1C} = \frac{A_1B_1}{B_1C}\,.$$

Therefore, as

$$\frac{BA}{BC} = \frac{A_1B_1}{B_1C} \quad \text{and} \quad \angle ABC = \angle A_1B_1C,$$

the triangles ΔBAC and ΔB_1A_1C are similar. This implies that

$$\angle B_1A_1C = \angle BAC = 60^\circ\,.$$

Similarly, we can show that $\angle BA_1C_1 = 60^\circ$. Therefore,

$$\angle BA_1C = \angle BA_1C_1 + \angle C_1A_1B_1 + \angle B_1A_1C = 60^\circ + 60^\circ + 60^\circ = 180^\circ\,.$$

This proves that B, A_1, C are colinear.

OC99. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ so that, for all $x \in \mathbb{Q}^+$ we have

$$f\left(\frac{x}{x+1}\right) = \frac{f(x)}{x+1} \text{ and } f\left(\frac{1}{x}\right) = \frac{f(x)}{x^3}\,.$$

Originally question 1 from Turkey Team Selection Test 2011.

Solved by Michel Bataille and we give his solution.

We show that the solutions are the functions $f_a : \mathbb{Q}^+ \to \mathbb{Q}^+$ (where $a \in \mathbb{Q}^+$) defined by

$$f_a(x) = a \cdot \frac{n^2}{d} \text{ if } x = \frac{n}{d} \text{ with } n, d \in \mathbb{N},\ \gcd(n,d) = 1\,.$$

Such a function f_a is a solution. Indeed, the equation $f_a\left(\frac{1}{x}\right) = \frac{f_a(x)}{x^3}$ is quickly verified and noticing that $\gcd(n, n+d) = 1$ if $\gcd(n,d) = 1$, we have

$$f_a\left(\frac{x}{x+1}\right) = f_a\left(\frac{n}{n+d}\right) = a \cdot \frac{n^2}{n+d} = \frac{a\cdot(n^2/d)}{1+\frac{n}{d}} = \frac{f_a(x)}{x+1}\,.$$

whenever $x = \frac{n}{d}$ with $n, d \in \mathbb{N},\ \gcd(n,d) = 1$.

Conversely, let f be an arbitrary solution and let $a = f(1)$. We show that $f\left(\frac{n}{d}\right) = f_a\left(\frac{n}{d}\right)$ for all $n, d \in \mathbb{N}$, $\gcd(n,d) = 1$ by induction on the positive integer $\max(n,d)$.

If $\max(n,d) = 1$, then $n = d = 1$ and $f(1) = a = a \cdot \frac{1^2}{1} = f_a(1)$.

Let $k \geq 2$ and assume that $f\left(\frac{\nu}{\delta}\right) = f_a\left(\frac{\nu}{\delta}\right)$ wherever $\nu, \delta \in \mathbb{N}$, $\gcd(\nu,\delta) = 1$ with $\max(\nu,\delta) < k$. Let $n, d \in \mathbb{N}$ $\gcd(n,d) = 1$ with $\max(n,d) = k$. Note that $n = d$ cannot occur, so we distinguish the two cases $n < d$ and $d < n$. In the former case, since $\gcd(n, d-n) = 1$ and $\max(n, d-n) < d = k$, we may write

$$f\left(\frac{n}{d}\right) = f\left(\frac{n/(d-n)}{1 + (n/(d-n))}\right) = \frac{f(n/(d-n))}{d/(d-n)} = \frac{an^2/(d-n)}{d/(d-n)} = a \cdot \frac{n^2}{d}.$$

In the latter case, using the previous case for the calculation of $f(d/n)$,

$$f\left(\frac{n}{d}\right) = f\left(\frac{1}{d/n}\right) = \frac{f(d/n)}{d^3/n^3} = a \cdot \frac{d^2}{n} \cdot \frac{n^3}{d^3} = a \cdot \frac{n^2}{d}\,.$$

In any event, $f\left(\frac{n}{d}\right) = a \cdot \frac{n^2}{d} = f_a\left(\frac{n}{d}\right)$ and the induction step is complete.

OC100. Let a_n be the sequence defined by

$$a_0 = 1,\, a_1 = -1, \text{ and } a_n = 6a_{n-1} + 5a_{n-2}\, \forall n \geq 2\,.$$

Prove that $a_{2012} - 2010$ is divisible by 2011.

Originally question 2 from Vietnam National Olympiad 2011, Day 2.

No solution was received to this problem. We give a solution by the editor .

Consider the quadratic equation

$$X^2 - 6X - 5 \equiv 0 \pmod{2011}\,.$$

The discriminant of this equation is $\Delta = 56$ which is a quadratic residue modulo 2011. [This can be seen either by calculating the Legendre symbol, or more easely by seeing that $56 \equiv 56 + 4 * 2011 = 8100 \pmod{2011}$.]

Therefore, as $p = 2011$ is a prime, the equation has two roots $s \not\equiv t \pmod{2011}$. Moreover, it is clear that neither of them is $0 \pmod{2011}$. It follows that s, t are invertible modulo 2011.

As the characteristic equation of the recursion $\pmod{2011}$ has two roots, and we work over a field, it follows that there exists some integers C_1, C_2 so that

$$a_n \equiv C_1 s^n + C_2 t^n \pmod{2011}\ ;\ \forall n\,.$$

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By Fermat Little Theorem, we have

$$s^{2010} \equiv t^{2010} \equiv 1 \pmod{2011}$$

Therefore

$$\begin{aligned} a_{2012} &\equiv C_1 s^{2012} + C_2 t^{2012} \equiv C_1 s^{2010} s^2 + C_2 t^{2010} t^2 \\ &\equiv C_1 s^2 + C_2 t^2 \equiv a_2 \pmod{2011} . \end{aligned}$$

As $a_2 = -6 + 5 = -1$, the claim of the problem follows.

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