Another Application of the Harmonic Mean

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These lines were motivated by the Problem Solver's Toolkit No. 6 ("The Harmonic Mean File" [2013 : 262 - 265]), where J. Chris Fisher provided constructions involving the harmonic mean. Here we see how the construction of a harmonic sequence can be modified to produce the reciprocals of the numbers in three familiar sequences that are defined by linear recurrences—the Fiboncci, Lucas, and Pell numbers.

In Figure 1, we recall one way to construct harmonic means.

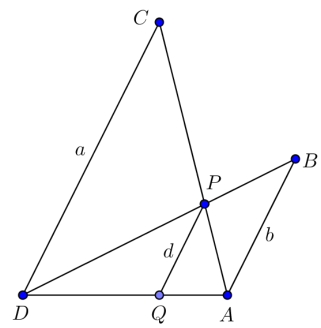

Figure 1: PQ is half the harmonic mean of CD and AB: $\frac{1}{d} = \frac{1}{a} + \frac{1}{b}$.

Specifically, let $ABCD$ be a trapezoid with $DC||AB$ and the diagonals intersecting in P; let the line through P that is parallel to those parallel sides meet AD in Q. If $a = DC$, $b = AB$, and $d = PQ$, then d is half the harmonic mean of the lengths a and b:

$$\frac{1}{d} = \frac{1}{a} + \frac{1}{b}.$$

The Fibonacci Numbers.

The nth term of the Fibonacci sequence $F_1, F_2, F_3, \ldots$ satisfies the linear recurrence, $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. To construct the reciprocals of these numbers, consider the rectangle $ABDA'$ in Figure 2, whose diagonals intersect in the point labeled D_3. For each point D_n with $n \geq 3$ on the diagonal AD we define A_n and B_n to be the projections of D_n on the sides AA' and AB, respectively. D_{n+1} can then be defined recursively to be the intersection of AD with A_nB_{n-1}. Starting with $A'D = f_1$, $AB = f_2$, and setting $f_3 = A_3D_3 = AB_3$, then we have $\frac{1}{f_3} = \frac{1}{f_2} + \frac{1}{f_1}$ (cf. Figure 1). With $f_n := A_nD_n = AB_n$ we get a sequence of segments having lengths f_n that satisfy

$$\frac{1}{f_n} = \frac{1}{f_{n-1}} + \frac{1}{f_{n-2}}.$$

Setting $f_1 = f_2 = 1$, we conclude that $\frac{1}{f_n} = F_n$ are elements of the Fibonacci sequence. In other words, $\{f_n\} = \left\{1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \dots\right\}$ is the sequence of reciprocals of the Fibonacci numbers.

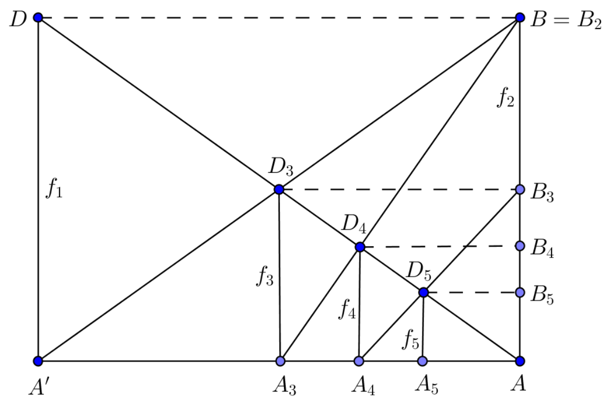

Figure 2: Sequence of segments whose lengths are reciprocals of the Fibonacci numbers.

The Lucas Numbers.

Of course, there was no reason to require $AB = A'D$ in the previous argument. We modify Figure 2 by allowing $\ell_1 = A'D$ to have a length different from that of $\ell_2 = AB$. Here $\ell_n = A_nD_n = AB_n$ satisfies

$$\frac{1}{\ell_n} = \frac{1}{\ell_{n-1}} + \frac{1}{\ell_{n-2}}.$$

Figure 3 shows the case $\ell_1 = \frac{1}{2}$ and $\ell_2 = 1$, so that $\{\ell_n\} = \left\{\frac{1}{2}, 1, \frac{1}{3}, \frac{1}{4}, \frac{1}{7}, \frac{1}{11}, \dots\right\}$. We recognize the elements of this sequence to be reciprocals of the Lucas numbers L_n: $L_1 = 2, L_2 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

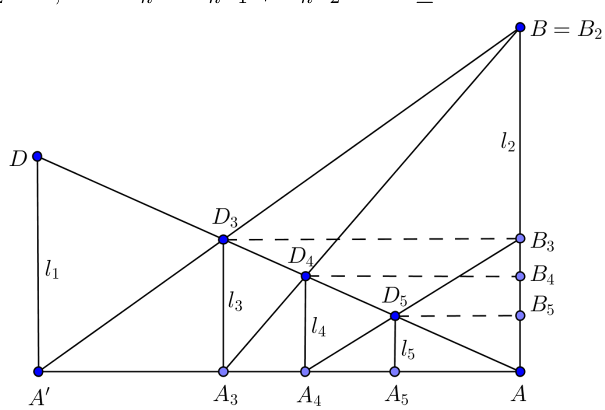

Figure 3: Sequence of segments whose lengths ℓ_n are reciprocals of the Lucas numbers.

The Pell Numbers.

The Pell numbers $\{P_n\} = \{1, 2, 5, 12, 29, \ldots\}$ satisfy the recursion $P_n = 2P_{n-1} + P_{n-2}$ with initial values $P_1 = 1$ and $P_2 = 2$. This sequence is closely related to the Pell equation $x^2 - 2y^2 = 1$. (See, for example, *The On-Line Encyclopedia of Integer Sequences*, www.oeis.org, sequence number A000129.)

Figure 4 on the next page shows a square AB_1CA_1 with D the midpoint of the side A_1C (and therefore $A_1D = \frac{1}{2}AA_1$). Compared to the previous constructions, the Pell sequence requires an extra step, namely, reflect the points B_n (which lie along the side AB_1) in the diagonal AC to the points A_n (which lie along the side AA_1). Note that the recursion begins with the segment $AA_3 = AB_3 = B'_3D_3$; we set $D_n = AD \cap A_{n-1}B_{n-2}$, while B_n and B'_n are the projections of D_n on the sides AB_1 and AA_1, respectively. The reflected image of the line D_nB_n in the diagonal AC intersects the line AD in the point labeled D'_n, and the base AA_1 in A_n. It follows that for $n \geq 3$,

$$B'_nD_n = AB_n = AA_n \quad \text{and} \quad A_nD'_n = \frac{1}{2}AA_n. \tag{1}$$

We begin with $D_3 = AD \cap A_2B_1$ so that B'_3D_3 is half the harmonic mean of $A_2D'_2$ and AB_1. But $B'_3D_3 = AB_3 = AA_3$, while $A_2D'_2 = \frac{1}{2}AA_2$, whence

$$\frac{1}{AA_3} = \frac{1}{B'_3D_3} = \frac{1}{A_2D'_2} + \frac{1}{AB_1} = 2\frac{1}{AA_2} + \frac{1}{AA_1}.$$

From (1) we see that in general,

$$\frac{1}{AA_n} = \frac{1}{B'_nD_n} = \frac{1}{A_{n-1}D'_{n-1}} + \frac{1}{AB_{n-2}} = 2\frac{1}{AA_{n-1}} + \frac{1}{AA_{n-2}}.$$

Setting $p_1 = AA_1 = 1$ and $p_2 = AA_2 = \frac{1}{2}$, we have $p_3 = AA_3 = \frac{1}{5}, p_4 = AA_4 = \frac{1}{12}$, and, in general, $\frac{1}{p_n} = 2\frac{1}{p_{n-1}} + \frac{1}{p_{n-2}}$. Define $P_n = \frac{1}{p_n}$ for $n \geq 1$. Then $P_1 = 1, P_2 = 2$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 3$, so that our constructed lengths p_n are reciprocals of the Pell numbers. Note, finally, that if D had been defined to be the point on the segment A_1C for which $A_1D = \frac{1}{k}A_1C$, then the numbers p_n would satisfy the more general Pell recursion $\frac{1}{p_n} = k\frac{1}{p_{n-1}} + \frac{1}{p_{n-2}}$.

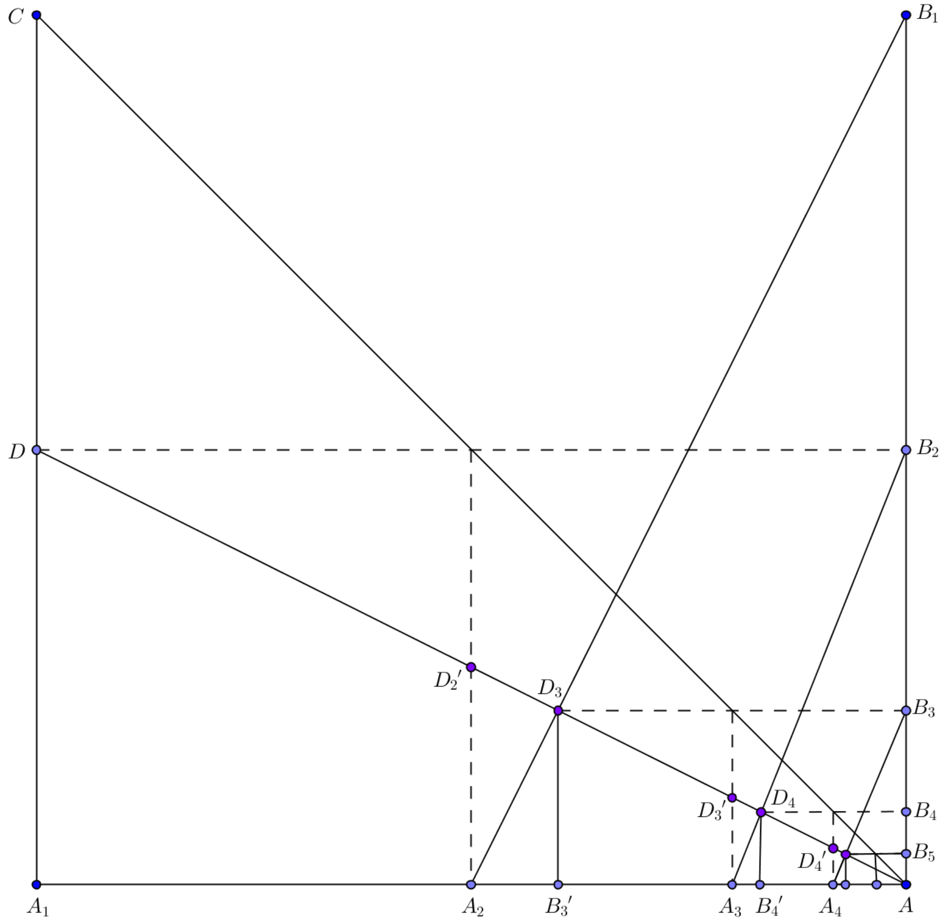

Figure 4: Sequence of segments whose lengths p_n are reciprocals of the Pell numbers.