CC100. In a 6 team tournament, each team played with each other team exactly once. A team gets 3 points for a victory, 1 point for a draw and 0 for a defeat. After the tournament, the sum of the scores by all the teams is 41. Prove that the exists a group of 4 teams where each team tied at least once.

CONTEST CORNER SOLUTIONS

CC46. Starting with the input (m, n), Machine A gives the output (n, m). Starting with the input (m, n), Machine B gives the output $(m + 3n, n)$. Starting with the input (m, n), Machine C gives the output $(m - 2n, n)$. Natalie starts with the pair $(0, 1)$ and inputs it into one of the machines. She takes the output and inputs it into any one of the machines. She continues to take the output that she receives and inputs it into any one of the machines. (For example, starting with $(0, 1)$, she could use machines B, B, A, C, B in that order to obtain the output $(7, 6)$.) Is it possible for her to obtain $(20132013, 20142014)$ after repeating this process any number of times ?

This problem was inspired by 2009 Fermat Contest, number 24.

Solved by G. Geupel ; R. Hess ; and S. Muralidharan. We present the solution by S. Muralidharan.

For any machine M and input (x, y) let us write $M(x, y)$ for the output. Given that

$$A(m, n) = (n, m)$$

$$B(m, n) = (m + 3n, n)$$

$$C(m, n) = (m - 2n, n)$$

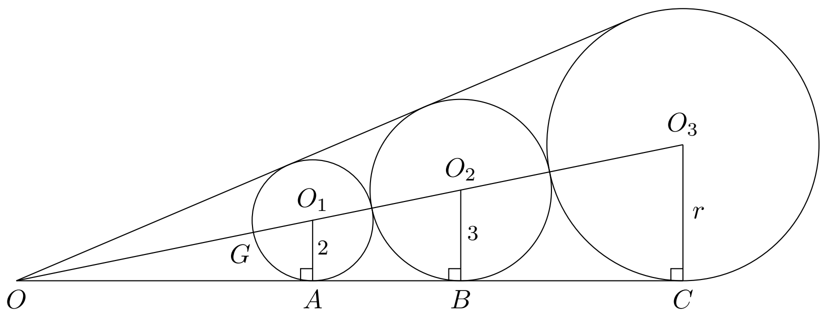
we clearly have $GCD(M(m, n)) = GCD(m, n)$ for any machine $M = A, B, C$ and any pair of integers (where GCD denotes the greatest common divisor). Since $GCD(0, 1) = 1$ and $GCD(20132013, 20142014) = 10001$, it follows that Natalie can not obtain $(20132013, 20142014)$ starting with $(0, 1)$.

CC47. A circle of radius 2 is tangent to both sides of an angle. A circle of radius 3 is tangent to the first circle and both sides of the angle. A third circle is tangent to the second circle and both sides of the angle. Find the radius of the third circle.

Originally 2005 W.J. Blundon Contest, problem 10.

Solved by Š. Arslanagić ; M. Bataille ; M. Coiculescu ; M. Amengual Covas ; G. Geupel ; J. G. Heuver ; S. Muralidharan ; and T. Zvonaru. We present solution by Matei Coiculescu with figure by Miguel Amengual Covas.

Denote by R the radius of the third circle. Let O be the vertex of the angle and let O_1, O_2 and O_3 be the centres of the respective circles with points of tangency A, B and C, respectively. The bisector OO_3 passes through the centres of the circles and intersects the circle with radius 2 at G. We denote the length OG by x.



It is easily seen that $\Delta OAO_1 \sim \Delta OBO_2 \sim \Delta OCO_3$. Thus $\frac{2}{3} = \frac{2+x}{7+x}$, or $x = 8$. Then $\frac{3}{r} = \frac{15}{18+r}$, which gives us $r = 4.5$.

CC48. Determine whether there exist two real numbers a and b such that both $(x-a)^3 + (x-b)^2 + x$ and $(x-b)^3 + (x-a)^2 + x$ contain only real roots.

Originally 2012 Sun Life Financial Repéchage Competition, problem 6.

Solved by M. Bataille ; R. Hess ; S. Muralidharan ; P. Perfetti ; and T. Zvonaru. We give a composite solution.

Suppose $(x-a)^3 + (x-b)^2 + x = x^3 - (3a-1)x^2 + (3a^2 - 2b + 1)x - (a^3 - b^2)$ has only real roots. Let r, s, t be these roots. Then

$$(x-r)(x-s)(x-t) = x^3 - (3a-1)x^2 + (3a^2 - 2b + 1)x - (a^3 - b^2).$$

Comparing coefficients we have

$$r + s + t = 3a - 1, \quad rs + st + tr = 3a^2 - 2b + 1.$$

Now observe that

$$\begin{aligned}(r+s+t)^2 &= r^2 + s^2 + t^2 + 2(rs + st + tr) \\ &= \left(\frac{r^2+s^2}{2} + \frac{s^2+t^2}{2} + \frac{t^2+r^2}{2}\right) + 2(rs + st + tr) \\ &\geq (rs + st + tr) + 2(rs + st + tr) \\ &= 3(rs + st + tr).\end{aligned}$$

Therefore,

$$(3a-1)^2 \geq 3(3a^2 - 2b + 1).$$

Rearranging this we get $6a - 6b \leq -2$. Similarly, since $(x-b)^3 + (x-a)^2 + x$ has only real roots, $6b - 6a \leq -2$. Adding these two inequalities we deduce $0 \leq -4$, a contradiction. Thus it is impossible for the two given polynomials to have only real roots.

Editor's Comment : Many submitted solutions involved analyzing the derivatives of the cubics.

CC49. Coins are placed on some of the 100 squares in a 10×10 grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal).

Originally 2009 University of Waterloo Big E Contest, Question 4.

One solution was received, which managed to place 34 coins on the board, which is not the minimum possible number.

CC50. Show that the square root of a natural number of five or fewer digits never has a decimal part starting 0.1111, but that there is an eight-digit number with this property.

Originally 2005 APICS Math Competition, Question 7.

One incorrect solution was received.