# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

**1464**. [1989 : 207 ; 1990 : 282-284] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle $ABC$, so that $A' \in BC$, $B' \in CA$, $C' \in AB$.

(a) Prove that

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'A} \tag{1}$$

if and only if the centroids $G, G'$ of the two triangles coincide.

(b) Prove that if (1) holds, and either the circumcenters $O, O'$ or the orthocenters $H, H'$ of the triangles coincide, then $\Delta ABC$ is equilateral.

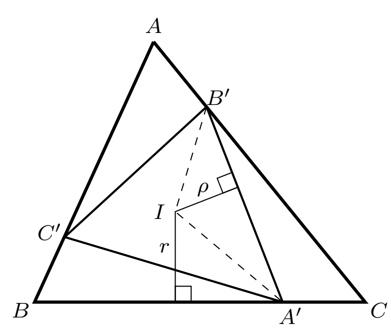
(c)★ If (1) holds and the incenters $I, I'$ of the triangles coincide, characterize $\Delta ABC$.

*Solution by C.R. Pranesachar, Indian Institute of Science, Bangalore, India.*

(c)★ We will show that, as in part (b), (1) together with $I = I'$ imply that $\Delta ABC$ (as well as $\Delta A'B'C'$) must be equilateral. Let

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'A} = w$$

be the common ratio into which the sides are divided by $A', B'$, and $C'$. Let, further, $BC = a, CA = b, AB = c$, area $[ABC] = F$, semiperimeter of triangle $ABC = s$, its inradius $= r$, and inradius of triangle $A'B'C' = \rho$.



By the Cosine formula we have

$$\begin{aligned} A'B'^2 &= A'C^2 + CB'^2 - 2\cdot A'C\cdot CB'\cdot\cos C \\ &= \frac{a^2}{(1+w)^2} + \frac{b^2w^2}{(1+w)^2} - \frac{2abw}{(1+w)^2}\cdot\frac{(a^2+b^2-c^2)}{2ab} \\ &= \frac{1}{(1+w)^2}\ \left(b^2(w^2-w)+c^2w+a^2(1-w)\right). \end{aligned}$$

Also

$$\begin{aligned} [IA'CB'] &= [IA'C]+[ICB'] = \frac12 r\cdot A'C + \frac12 r\cdot CB' \\ &= \frac12\cdot\frac{F}{s}\cdot\frac{(a+bw)}{(1+w)} = \frac{(a+bw)F}{(a+b+c)(1+w)}; \end{aligned}$$

and

$$[B'A'C] = \frac12\cdot A'C\cdot CB'\cdot\sin C = \frac12\frac{abw}{(1+w)^2}\cdot\frac{2F}{ab} = \frac{wF}{(1+w)^2}.$$

Hence

$$\begin{aligned} [IA'B'] &= [IA'CB'] - [B'A'C] \\ &= \frac{(a+bw)F}{(a+b+c)(1+w)} - \frac{wF}{(1+w)^2} = \frac{F}{(1+w)^2}\cdot\frac{(bw^2-cw+a)}{(a+b+c)}. \end{aligned}$$

Now the inradius $\rho$ of triangle $A'B'C'$ is given by

$$\rho = 2\cdot\frac{[IA'B']}{A'B'};$$

so

$$\rho^2 = 4\cdot\frac{[IA'B']^2}{A'B'^2} = \frac{4F^2}{(1+w)^2(a+b+c)^2}\cdot\frac{(bw^2-cw+a)^2}{(b^2(w^2-w)+c^2w+a^2(1-w))}\ .$$

We have two further expressions for $\rho^2$ obtained by cyclically permuting $a, b, c$ in the last expression. Canceling the common factor $4F^2/((1+w)^2(a+b+c)^2)$, we have

$$\begin{aligned} \frac{(aw^2-bw+c)^2}{(a^2(w^2-w)+b^2w+c^2(1-w))} &= \frac{(bw^2-cw+a)^2}{(b^2(w^2-w)+c^2w+a^2(1-w))} \\ &= \frac{(cw^2-aw+b)^2}{(c^2(w^2-w)+a^2w+b^2(1-w))}\ . \end{aligned}$$

From the equality of the second and third quotients we get (after cross-multiplying and removing the nonzero factor $w(w^2-w+1)(a+b+c)$),

$$\begin{aligned} (b^3+c^3-b^2c+bc^2-ab^2-ac^2)w^2 - 2ca(c-a)w \\ -(a^3+b^3+a^2b-ab^2-a^2c-b^2c) = 0. \quad (2) \end{aligned}$$

Similarly from the equality of the first and third quotients we get

$$(c^3 + a^3 - c^2a + ca^2 - bc^2 - ba^2)w^2 - 2ab(a - b)w \\ - (b^3 + c^3 + b^2c - bc^2 - b^2a - c^2a) = 0. \quad (3)$$

We know that if two quadratic equations $P_1t^2 + Q_1t + R_1 = 0$ and $P_2t^2 + Q_2t + R_2 = 0$ have a common root then

$$(R_1P_2 - R_2P_1)^2 = (Q_1R_2 - Q_2R_1)(P_1Q_2 - P_2Q_1).$$

Thus, eliminating $w$ from the quadratic equations (2) and (3), we get on factoring

$$(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab) \\ \times (a^3 + b^3 + c^3 + abc)(a^3 + b^3 + c^3 - 2ab^2 - 2ac^2 + abc)^2 = 0.$$

Since the first and third factors on the left are positive, one of the remaining factors must be zero. From the Cauchy-Schwarz inequality applied to the vectors $(a, b, c)$ and $(b, c, a)$, we know that $a^2+b^2+c^2-bc-ca-ab = 0$ implies $a = b = c$. Finally, if $a^3+b^3+c^3-2ab^2-2ac^2+abc = 0$, we observe that had we chosen other pairs of equations in the previous argument we would have cyclically permuted the roles of $a, b, c$ in equations (2) and (3), resulting in $a^3 + b^3 + c^3 - 2bc^2 - 2ba^2 + abc = 0$ and $a^3 + b^3 + c^3 - 2ca^2 - 2cb^2 + abc = 0$. From these three equations we deduce that

$$a(b^2 + c^2) = b(c^2 + a^2) = c(a^2 + b^2),$$

from which we again conclude that $a = b = c$.

*No other solutions have been received.*

*Pranesachar used MAPLE for his computations, but the results can be verified easily enough by hand. He remarked that because his calculations used squares of distances, his argument shows further that I could not even be an excentre of triangle $A'B'C'$. He went on to prove that even if the ratio w were allowed to be negative (in which case $A', B'$, and $C'$ lie outside triangle ABC), the incentres of the two triangles would coincide only if the triangles were equilateral. Because the proof of this claim is similar to the above solution, we will not reproduce it here. This result is in contrast with Problem 1492(b) [1999 : 508-510], where he with his coauthor B.V. Venkatachala showed that there exist nonequilateral triangles for which the incentre of $\Delta ABC$ coincides with an excentre of $\Delta A'B'C'$ when condition (1) is replaced by the requirement that $AA' = BB' = CC'$.*

**3701**. [2012 : 23, 25] *Proposed by R. F. Stöckli, Buenos Aires, Argentina.*

Let $S$ be the set of all real continuous functions $f$ defined on the closed interval $[0, 1]$ with $f(0) = f(1) = 0$. Find all numbers $0 < r < 1$ such that for every $f$ belonging to $S$ there exists $c = c(f)$ and $d = d(f)$ belonging to $[0, 1]$ such that $d - c = r$ and $f(c) = f(d)$.

*I. Solution by George Apostolopoulos, Messolonghi, Greece.*

The values of $r$ are reciprocals of positive integers exceeding 1.

For integers $n \geq 2$, let $f(x)$ satisfy the conditions of the problem and let

$$g(x) = f\left(x + \frac{1}{n}\right) - f(x)$$

for $0 \le x \le (n-1)/n$. Suppose that $g(x)$ never vanishes. Then $g(x)$ must be either always positive or always negative. But then

$$\begin{aligned} 0 < \sum_{i=0}^{n-1} \left| g\left(\frac{i}{m}\right) \right| &= \sum_{i=0}^{n-1} \left| f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right) \right| \\ &= \left| \sum_{i=0}^{n-1} f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right) \right| = |f(1) - f(0)| = 0, \end{aligned}$$

a contradiction. Therefore $f(x+1/n) = f(x)$ for some $x \in [0, 1-1/n]$ and $r = 1/n$ satisfies the condition of the problem.

However, we give a counterexample for any other value of $r \in (0,1)$. Suppose that $k = 1 - r\lfloor 1/r \rfloor$. Since $0 < 1/r - \lfloor 1/r \rfloor < 1$, it follows that

$$0 < k = 1 - r\lfloor 1/r \rfloor < r.$$

Define $f(0) = 0$, $f(k) = -\lfloor 1/r \rfloor$ and $f(r) = 1$. Extend $f$ linearly to the intervals $[0,k]$ and $[k,r]$. For $0 \le x \le 1-r$, let $f(x+r) = f(x)+1$. It is clear that $f(x+r)$ never equals $f(x)$. In addition,

$$f(1) = f\left(k + \left\lfloor \frac{1}{r} \right\rfloor r\right) = f(k) + \left\lfloor \frac{1}{r} \right\rfloor = 0.$$

*II. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.*

First let $r = 1/n$ for a positive integer $n$ exceeding 1. Define $g(x)$ as in the foregoing solution. Then for $f(x)$ satisfying the conditions of the problem, $g(0) + g(1/n) + g(2/n) + \cdots + g((n-1)/n) = f(1) - f(0) = 0$. Hence either all of the $g(i/n)$ must vanish, or there are indices $i < j$ for which $g(i/n)$ and $g(j/n)$ have opposite signs. In the latter case, since $f(x)$ is continuous, there exists $c \in [i/n, j/n]$ for which $f(c+1/n) - f(c) = 0$.

On the other hand, let $r \in (0,1)$ not be the reciprocal of an integer. For $0 \le x \le 1$, define

$$f(x) = \left(\sin \frac{\pi x}{r}\right)^2 - x\left(\sin \frac{\pi}{r}\right)^2.$$

Then $f(x)$ is continuous, $f(1) = f(0) = 0$ and

$$\begin{aligned} f(x+r) &= \left(\sin \frac{\pi(x+r)}{r}\right)^2 - (x+r)\left(\sin \frac{\pi}{r}\right)^2 \\ &= \left(\sin\left(\frac{\pi x}{r} + \pi\right)\right)^2 - (x+r)\left(\sin \frac{\pi}{r}\right)^2 \\ &= \left(\sin\left(\frac{\pi x}{r}\right)\right)^2 - x\left(\sin \frac{\pi}{r}\right)^2 - r\left(\sin \frac{\pi}{r}\right)^2 = f(x) - r\left(\sin \frac{\pi}{r}\right)^2. \end{aligned}$$

Since $\sin(\pi/r) \ne 0$, $f(x+r) \ne f(x)$ for all $x \in [0, 1-r]$.

*Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer. Geupel provided a piecewise linear example for the case $1/r$ not an integer. Bataille gave the same example as in the second solution. There was one incorrect solution.*

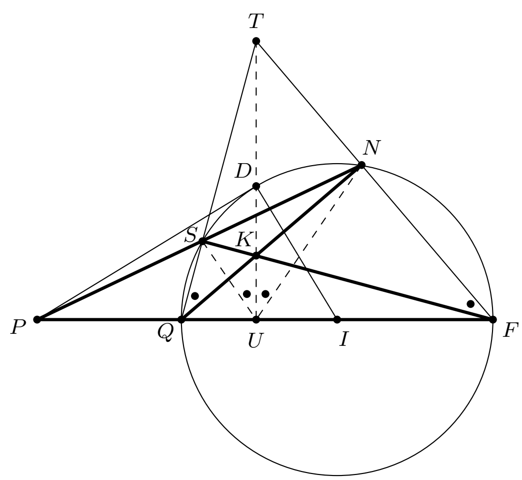
**3702**. [2012 : 23, 25] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let the incircle (with centre $I$) of triangle $ABC$ touch the sides $BC$ at $D$, $CA$ at $E$, and $AB$ at $F$. Define $M$ and $P$ to be the points where $BC$ intersects the lines $IE$ and $IF$, $N$ and $Q$ to be the second intersections of the incircle with $IE$ and $IF$, and $R$ and $S$ to be the second intersections of the incircle with $MQ$ and $PN$. If $H$ and $K$ are the points where $NQ$ intersects $ER$ and $FS$, prove that $\angle KDH = \angle BAC$.

*I. Solution by the proposer.*

We will prove that $DK$ is perpendicular to $IF$. Since $IF$ is, by definition, also perpendicular to $AB$, we will be able to deduce that $KD||BA$. A similar argument will give us $DH||AC$, so that the angles $KDH$ and $BAC$ have corresponding sides parallel and are therefore equal as claimed.

Consider the portion of the given configuration shown in the accompanying figure : the diameter $FQ$ lies along the line $PI$, while the secant $SN$ also passes through $P$. The line $PD$ is tangent at $D$ to the circle with centre $I$, and the chords $SF$ and $QN$ intersect at $K$. Define the further points $T = QS \cap NF$ and $U = TK \cap QF$. We are to prove that $TK$ is perpendicular to $QF$ at $U$ and passes through $D$.




Note that because $QF$ is a diameter of a circle through $S$ and $N$, $K$ is the orthocentre of triangle $QFT$; it follows that $TK$ defines the third altitude, whence $TU \perp QF$. It remains to show that $D$ lies on $TK$. To that end we note that the quadrilateral $SQUK$ (with right angles at $S$ and $U$) is cyclic, so that $\angle SQK = \angle SUK$. Similarly, the cyclic quadrilateral $FNKU$ gives us $\angle KFN = \angle KUN$. We have, therefore,

$$\begin{aligned}\angle SUN &= \angle SUK + \angle KUN = \angle SQK + \angle KFN \\ &= \angle SQN + \angle SFN = 2\angle SQN = \angle SIN.\end{aligned}$$

Thus, $SUIN$ is a cyclic quadrilateral and, therefore, $PS \cdot PN = PU \cdot PI$. But $PD^2 = PS \cdot PN$, whence $PD^2 = PU \cdot PI$. Since $PDI$ is a right-angled triangle, we deduce that $DU$ is perpendicular to $PI$. In other words, $D$ lies on the perpendicular to $PI$ through $U$, as do $T$ and $K$. We conclude that the lines $DK$ and $PI(= IF)$ are perpendicular, as desired. A similar argument using $M$ in place of $P$ finishes the proof.

*II. Composite of solutions by Václav Konečný, Big Rapids, MI, USA ; and by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Starting with the cyclic quadrilateral $FQSN$ whose side $FQ$ is the diameter of the incircle $\Gamma$, point $K$ is the intersection of its diagonals while the external point $P$ is the intersection of two of its sides. The tangent to $\Gamma$ from $P$ touches it at $D$ and, therefore, $DK$ is the polar of $P$ ; furthermore, the point where $DK$ meets $\Gamma$ again, call it $D'$, must be the point where the second tangent from $P$ touches $\Gamma$. Consequently, $PI$ is the perpendicular bisector of $DD'$ and, since $F$ is also on this line, we conclude that $DK \perp IF$. But $IF \perp BA$, so $KD||BA$. Similarly, $H$ lies on the polar of $M$ and therefore $DH \perp IE$, or $DH||AC$. Thus, $\angle KDH = \angle BAC$, as desired.

*Also solved by MICHEL BATAILLE, Rouen, France ; OLIVER GEUPEL, Brühl, NRW, Germany ; and EDMUND SWYLAN, Riga, Latvia.*

*The projective theorem used in solution II applies to any cyclic quadrilateral $FQSN$, whether or not one of its sides is a diameter. Specifically, if $FQSN$ is a convex quadrilateral inscribed in a circle with centre $I$, and the points $K, P$, and $D$ are related to the quadrilateral as in the problem, then $DK \perp PI$. As far as this editor can see, none of the submitted solutions could be easily modified to provide an elementary Euclidean proof of this more general result.*

**3703**. [2012 : 24, 26] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let $a$, $b$, and $c$ be the sides, and $r$ the inradius of a triangle $ABC$. Prove that

$$\frac{a\sqrt[3]{abc}}{bc(a^2+bc)} + \frac{b\sqrt[3]{abc}}{ac(b^2+ca)} + \frac{c\sqrt[3]{abc}}{ab(c^2+ab)} \le \frac{1}{8r^2}\,.$$

*Solution by John G. Heuver, Grande Prairie, AB.*

Let $H$ denote the summation on the left side of the given inequality. Then

$$H = \frac{\sqrt[3]{abc}}{abc}\left(\frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab}\right)\,. \qquad (1)$$

In the solution to ***Crux*** problem **3374** [2009 : 414–415] Peter Y. Woo proved that

$$\frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab} \le \frac{a+b+c}{2\sqrt[3]{abc}}\,. \qquad (2)$$

Let $s$ and $R$ denote the semi-perimeter and circumradius of triangle $ABC$, respectively. It is well known that $abc = 4rsR$ which, when combined with (1) and (2) then yields

$$H \le \frac{a+b+c}{2abc} = \frac{2s}{8rsR} = \frac{1}{4rR} \le \frac{1}{8r^2}$$

since $2r \le R$ by Euler's Inequality.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina(two solutions); RADOUAN BOUKHARFANE, Polytechnique de Montréal, PQ; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC(two solutions); PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; EDMUND SWYLAN, Riga, Latvia; and the proposer. There was also one incomplete solution.*

**3704**. [2012 : 24, 26] *Proposed by Richard McIntosh, University of Regina, Regina, SK.*

Let $p \equiv 1 \pmod 3$ be a prime and let $n$ be an integer satisfying $n^2+n+1 \equiv 0 \pmod p$. Prove that $(n+1)^p \equiv n^p + 1 \pmod{p^3}$.

*Composite of similar solution by George Apostolopoulos, Messolonghi, Greece; and Michel Bataille, Rouen, France.*

Let $Z[x]$ denote the ring of all polynomials in $x$ with integer coefficients. Since $p \mid n^2+n+1$ implies $p^2 \mid (n^2+n+1)^2$, it suffices to prove that

$$p(x^2+x+1)^2 \mid (x+1)^p - x^p - 1$$

in $Z[x]$.

Let $f(x) = x^2+x+1$ and $g(x) = (x+1)^p - x^p - 1$. Since $p \mid \binom{p}{k}$ for all $k = 1, 2, \ldots, p-1$ and $g(x) = \sum_{k=1}^{p-1} \binom{p}{k} x^{p-k}$, clearly $p \mid g(x)$. It now suffices to show that the two complex roots, $\omega$ and $\overline{\omega}$, of $f(x)$, where $\omega = \frac{-1+\sqrt{3}i}{2}$, are multiple roots of $g(x)$.

Since $\omega^2+\omega+1 = 0$, $\omega^3 = 1$ and $p \equiv 1 \pmod 3$ we have $\omega^p = \omega$. Hence,

$$g(\omega) = (\omega+1)^p - \omega^p - 1 = (-\omega^2)^p - \omega^p - 1 = -\omega^2 - \omega - 1 = 0\,.$$

Furthermore, $g'(x) = p(x+1)^{p-1} - px^{p-1}$ so

$$g'(\omega) = p((\omega+1)^{p-1} - \omega^{p-1}) = p((-\omega^2)^{p-1} - \omega^{p-1}) = p(1-1) = 0\,.$$

Therefore, both $\omega$ and $\overline{\omega}$ are multiple roots of $g(x)$; that is, $(f(x))^2 \mid g(x)$ and our proof is complete.

*Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There was also a partially incorrect solution which claims that the condition $p \equiv 1 \pmod 3$ follows from the other given condition. The example $n = 1$ and $p = 3$ shows that the claim is false. However, it is interesting to note that using Fermat's little theorem one can easily show that if $(n+1)^p \equiv n^p + 1 \pmod p$ then $p \equiv 1 \pmod 3$ or $p = 3$.*

**3705**. [2012 : 24, 26] *Proposed by Michel Bataille, Rouen, France.*

Let $ABC$ be a scalene triangle and $G$ its centre of gravity. Let the perpendicular to $BC$ through $G$ meet the internal bisector of $\angle BAC$ at $A'$.

(a) Show that $G$ and the orthogonal projections of $A'$ onto the lines $AB$ and $AC$ are collinear.

(b) If $B'$ and $C'$ are defined similarly to $A'$, prove that

$$\frac{GA' \cdot GB'}{CA \cdot CB} + \frac{GB' \cdot GC'}{AB \cdot AC} + \frac{GC' \cdot GA'}{BC \cdot BA} = \frac{1}{9}\,.$$

*Similar solutions by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

(a) Define $A_2$ to be the point where $AA'$ again intersects the circumcircle of triangle $ABC$. Denote the midpoint of side $BC$ by $A_1$, and note that $A_1A_2 \perp BC$. Because $GA'$ has been defined to be perpendicular to $BC$, we have $A_1A_2 || GA'$, whence triangles $AA'G$ and $AA_2A_1$ are homothetic with ratio of magnitude $\frac{2}{3}$. If $A_1, B_1, C_1$ are the feet of perpendiculars dropped from $A_2$ to $BC, CA, AB$, then these points lie on a line, namely the Simson line of $A_2$ with respect to $\Delta ABC$. If $D$ and $E$ are the orthogonal projections of $A'$ onto the lines $AC$ and $AB$, then the dilatation with centre $A$ and ratio $\frac{2}{3}$ that shrinks $\Delta AA_2A_1$ to $\Delta AA'G$ takes $B_1$ to $D$ and $C_1$ to $E$; consequently, $D, E, G$ are collinear.

(b) From the right triangle $BA_2A_1$ we see that $A_1A_2 = \frac{BC}{2} \tan \frac{A}{2}$; from the dilatation of part (a) we have $GA' = \frac{2}{3} A_1A_2$. Thus,

$$\frac{GA'}{BC} = \frac{1}{3} \tan \frac{A}{2}.$$

Multiplying this by the analogous quotient with respect to the point $B'$ and side $CA$, we have

$$\frac{GA' \cdot GB'}{CA \cdot CB} = \frac{1}{9} \tan \frac{A}{2} \tan \frac{B}{2}.$$

Continuing in this manner, we deduce that

$$\frac{GA' \cdot GB'}{CA \cdot CB} + \frac{GB' \cdot GC'}{AB \cdot AC} + \frac{GC' \cdot GA'}{BC \cdot BA} = \frac{1}{9}\left(\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2}\right).$$

The sum in parentheses on the right is known to be 1 (this is just the tangent-of-a-sum identity applied to three angles that sum to $90^\circ$), which completes the proof of part (b).

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; EDMUND SWYLAN, Riga, Latvia;*

*TITU ZVONARU, Comăneşti, Romania ; and the proposer. There was one incomplete submission.*

*Konečný observed that the role of $G$ can be replaced by any point $M$ (except $A$) on the median $AA_1$ ; specifically, the featured solution to part (a) shows that if $A'$ were defined to be the intersection of the bisector of $\angle BAC$ with the perpendicular to $BC$ through $M$, then $M$ is collinear with the orthogonal projections of $A'$ onto the lines $AB$ and $AC$.*

**3706**. [2012 : 24, 26] *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Prove that for all positive real numbers $a$, $b$, $c$, and $d$ which satisfy $a, b, c \geq 1$ and $abcd = 1$,

$$\sum_{\text{cyclic}} \frac{1}{(a^2 - a + 1)^2} \ \leq \ 4\,.$$

*I. Solution by Michel Bataille, Rouen, France.*

Since $2(x^2 - x + 1)^2 - (x^4 + 1) = (x - 1)^4 \geq 0$ for all real $x$, it suffices to show that

$$\frac{1}{a^4+1} + \frac{1}{b^4+1} + \frac{1}{c^4+1} + \frac{(abc)^4}{(abc)^4+1} \leq 2$$

or, equivalently,

$$\frac{1}{a^4+1} + \frac{1}{b^4+1} + \frac{1}{c^4+1} \leq 1 + \frac{1}{(abc)^4+1}.$$

Thus, we have to establish that

$$f(u) + f(v) + f(w) \leq f(u + v + w)$$

where $u = \ln a$, $v = \ln b$, $w = \ln c$, and

$$f(x) = \frac{1}{e^{4x}+1} - \frac{1}{2}.$$

It is straightforward to check that $f(0) = 0$, $f'(x) < 0$ and $f''(x) > 0$, so that $f(x)$ is negative, decreasing and convex on $(0, \infty)$. Thus $f'(x)$ increases to 0. For each fixed $y \geq 0$, let $\phi(x) = f(x+y) - f(x) - f(y)$. Then $\phi(0) = 0$ and the derivative $\phi'(x) = f'(x + y) - f'(x)$ is positive on $(0, \infty)$, so that $f(x + y) \geq f(x) + f(y)$. Therefore

$$f(u + v + w) \geq f(u + v) + f(w) \geq f(u) + f(v) + f(w)$$

and the desired inequality follows.

*II. Solution by the proposer.*

First we establish that, for $x, y \geq 1$,

$$\frac{1}{g(x)^2} + \frac{1}{g(y)^2} \leq 1 + \frac{1}{g(xy)^2},$$

where $g(t) = t^2 - t + 1$. The difference between the two sides of this inequality is

$$\left(1 - \frac{1}{g(x)^2}\right)\left(1 - \frac{1}{g(y)^2}\right) + \frac{1}{g(xy)^2} - \frac{1}{g(x)^2 g(y)^2},$$

a fraction whose denominator is $g(x)^2 g(y)^2 g(xy)^2$ and whose numerator is

$$\begin{aligned}
&(g(x)-1)(g(x)+1)(g(y)-1)(g(y)+1)g(xy)^2 \\
&\qquad + (g(x)g(y) - g(xy))(g(x)g(y) + g(xy)) \\
&= x(x-1)(x^2-x+2)y(y-1)(y^2-y+2)(x^2y^2-xy+1)^2 \\
&\qquad - (x-1)(y-1)(x+y)(2x^2y^2 - xy(x+y) + x^2 + y^2 - x - y + 2). \\
&= (x-1)(y-1)[xy(x^2-x+2)(y^2-y+2)(x^2y^2-xy+1)^2 \\
&\qquad - (x+y)(2x^2y^2 - xy(x+y) + x^2 + y^2 - x - y + 2)].
\end{aligned}$$

Since

$$(x^2 - x + 2)(y^2 - y + 2)(x^2y^2 - xy + 1) \geq 4$$

and

$$4xy = 2(x+y) + (2x-1)(2y-1) - 1 \geq 2(x+y),$$

this numerator is not less than

$$\begin{aligned}
&(x-1)(y-1)(x+y) \\
&\times[2(x^2y^2 - xy + 1) - 2x^2y^2 + xy(x+y) - (x+y)^2 + 2xy + (x+y) - 2] \\
&\qquad = (x-1)(y-1)(x+y)[xy(x+y) - (x+y)^2 + (x+y)] \\
&\qquad = (x-1)^2(y-1)^2(x+y)^2 \geq 0.
\end{aligned}$$

Hence, for $a, b, c \geq 1$ and $d = (abc)^{-1}$,

$$\begin{aligned}
\frac{1}{g(a)^2} + \frac{1}{g(b)^2} + \frac{1}{g(c)^2} &\leq 1 + \frac{1}{g(ab)^2} + \frac{1}{g(c)^2} \\
&= 2 + \frac{1}{g(abc)^2} = 2 + \frac{1}{g(1/d)^2} \\
&= 2 + \frac{d^4}{g(d)^2} \leq 4 - \frac{1}{g(d)^2},
\end{aligned}$$

which is the desired result. The final inequality is due to the fact that

$$2 - \frac{1 + d^4}{g(d)^2} = \frac{(1-d)^4}{g(d)^2} \geq 0.$$

*III. Solution by Kee-Wai Lau, Hong Kong, China.*

We prove that

$$\sum_{k=1}^{n} \frac{1}{(x_k^2 - x_k + 1)^2} \leq n$$

subject to the restrictions $x_1, x_2, \cdots, x_{n-1} \geq 1$ and $x_1 x_2 \cdots x_n = 1$. This holds for $n = 2$.

Denote the left side by $f_n(x_1, x_2, \cdots, x_{n-1})$ with $x_n$ dependent on the remaining variables. Assume as an induction hypothesis that

$$f_n(x_1, x_2, \cdots, x_{n-1}) \leq n$$

subject to the stated conditions. Assume that $x_1, x_2, \cdots, x_n \geq 1$, that $x_1 x_2 \cdots x_n x_{n+1} = 1$ and $f_{n+1}(x_1, \cdots, x_n)$ is the analogous left side of the inequality. Then

$$f_{n+1}(x_1, x_2, \cdots, x_{n-1}, 1) = f_n(x_1, x_2, \cdots, x_{n-1}) + 1 \leq n + 1.$$

It remains to show that

$$f_{n+1}(x_1, x_2, \cdots, x_{n-1}, x_n) \leq f_{n+1}(x_1, x_2, \cdots, x_{n-1}, 1)$$

when $x_1, \cdots, x_n \geq 1$. Now

$$\frac{\partial f_{n+1}}{\partial x_n} = \frac{2(1 - 2x_n)}{(x_n^2 - x_n + 1)^3} + \frac{2}{x_n} g(x_1 x_2 \cdots x_n)$$

where $g(t) = t^4(2-t)(t^2-t+1)^{-3}$. When $x_1 x_2 \cdots x_n \geq 2$, then $g(x_1, x_2, \cdots x_n) \leq 0$ and $\frac{\partial f_{n+1}}{\partial x_n} \leq 0$.

Now suppose that $1 \leq x_1 x_2 \cdots x_n \leq 2$. Since

$$g'(t) = t^3(t-1)(t^2 - t - 8)(t^2 - t + 1)^{-4} < 0$$

for $1 < t < 2$, then $g(t)$ is decreasing. Since $1 \leq x_n \leq x_1 x_2 \cdots x_n \leq 2$, it follows that $g(x_n) \geq g(x_1 x_2 \cdots x_n)$ and

$$\begin{aligned} \frac{\partial f_{n+1}}{\partial x_n} &\leq \frac{2(1 - 2x_n)}{(x_n^2 - x_n + 1)^3} + \frac{2x_n^3(2 - x_n)}{(x_n^2 - x_n + 1)^3} \\ &= \frac{-2(x_n + 1)(x_n - 1)^3}{(x_n^2 - x_n + 1)^3} < 0 \end{aligned}$$

for $x_n > 1$, so that $f_n(x_1, \cdots, x_n)$ is a decreasing function of $x_n$ and the desired result follows.

*IV. Solution by Haohao Wang and Yanping Xia, Southeast Missouri State University, Cape Girardeau, MO, USA.*

Define $f_n(x_1, x_2, \cdots, x_{n-1})$ as in the foregoing solution with $x_i \geq 1$ $(1 \leq i \leq n-1)$ and $x_n = (x_1 x_2 \cdots x_{n-1})^{-1}$. We have to show that $f_n$ is maximized at $(1, 1, \cdots, 1)$, where it assumes the value $n$.

A critical value $f_n$ occurs in the interior of its domain if and only if

$$0 = \frac{\partial f_n}{\partial x_i} = \frac{2}{x_i}[h(x_n) - h(x_i)]$$

or $h(x_n) = h(x_i)$ for $1 \le i \le n-1$, where $h(t) = t(2t-1)(t^2-t+1)^{-3}$. Since, for $t > 1$, $h'(t) = -(t-1)(8t^2+t-1)(t^2-t+1)^{-4} < 0$, $h$ is strictly decreasing on $[1,\infty)$ so the condition for an extremum is $x_1 = x_2 = \cdots = x_n$. This implies that $h(x_1) = h(x_1^{-(n-1)})$. However, $h(t) > h(t^{-1})$ when $t > 1$, so that $h(x_1) > h(x_1^{n-1}) > h(x_1^{-(n-1)})$ from which we see that there are no extrema in the interior of the domain of $f$.

Consider the values of $f_n$ on the boundary of its domain, say where $x_{n-1} = 1$. Then $f_n(x_1, x_2, \cdots, x_{n-2}, 1) = f_{n-1}(x_1, x_2, \cdots, x_{n-2}) + 1$ with $x_1, x_2, \cdots, x_{n-2} \ge 1$ and $x_1 x_2 \cdots x_{n-1} = 1$. By descent, we eventually need to prove the result for $n = 2$. But this follows from

$$\frac{1}{(x_1^2 - x_1 + 1)^2} + \frac{1}{(x_1^{-2} - x_1^{-1} + 1)^2} = 2 - \frac{(1-x_1)^4}{(x_1^2 - x_1 + 1)^2},$$

with equality if and only if $x_1 = x_2 = 1$. The desired result follows.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RADOUAN BOUKHARFANE, Polytechnique de Montréal, PQ; and OLIVER GEUPEL, Brühl, NRW, Germany. Geupel pointed out that the problem was previously posed by the proposer in Vietnamese and reproduced his solution. He refers us to Problem 1.45 on page 38f in the e-paper toanhocmuonmaumain.pdf, available on the MathLinks forum website www.mathlinks.ro/viewtopic.php?t=197674. There was one incorrect solution.*

*It is straightforward to adapt the first two solutions to establish generalizations of the last two.*

**3707**. [2012 : 24, 26] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k$ and $m$ be positive integers. Prove that

$$\int_0^\infty \frac{\sin^{2k} x}{(\pi^2 - x^2)((2\pi)^2 - x^2)\cdots((k\pi)^2 - x^2)} dx = 0\,.$$

*I. Solution by Michel Bataille, Rouen, France.*

Let $I$ be the given integral. Then I $= \int_0^\infty f(x)dx$ where $f(x) = 0$ when $x$ is a positive multiple of $\pi$ and $f(x) = (\sin^{2k} x)(\prod_{j=1}^k ((j\pi)^2 - x^2)^{-1}$ otherwise for $x \ge 0$. For $1 \le j \le k$,

$$\frac{\sin^{2k} x}{j^2\pi^2 - x^2} = \frac{\sin^{2k-1} x}{j\pi + x} \cdot \frac{\sin(j\pi - x)}{j\pi - x} \cdot (-1)^{j-1}$$

and $\lim_{x\to j\pi}(\sin(j\pi - x))(j\pi - x)^{-1} = 1$, so that $\lim_{x\to j\pi} f(x) = 0$ and $f(x)$ is continuous on $[0,\infty)$. Thus $\int_0^{(k+1)\pi} f(x)dx$ exists. Since also $|f(x)| \le (\prod_{j=1}^k |(j\pi)^2 - x^2)|^{-1}$ for $x \ge (k+1)\pi$ and $\int_{(k+1)\pi}^\infty (\prod_{j=1}^k |(j\pi)^2 - x^2|)^{-1} dx < \infty$, the integral $I$ exists.

We have the partial fraction decomposition

$$\frac{1}{\prod_{j=1}^k (j\pi)^2 - x^2} = a_j\left(\frac{1}{j\pi + x} + \frac{1}{j\pi - x}\right),$$

so that it suffices to prove that

$$\int_0^\infty \left( \frac{\sin^{2k} x}{j\pi + x} + \frac{\sin^{2k} x}{j\pi - x} \right) dx = 0.$$

Recall that $\int_0^\infty (\sin^{2k} x)x^{-1} dx$ exists. Then

$$\begin{aligned} \int_0^\infty \left( \frac{\sin^{2k} x}{j\pi + x} + \frac{\sin^{2k} x}{j\pi - x} \right) dx &= \int_0^\infty \frac{\sin^{2k} x}{j\pi + x} dx + \int_0^{j\pi} \frac{\sin^{2k} x}{j\pi - x} dx - \int_{j\pi}^\infty \frac{\sin^{2k} x}{x - j\pi} dx \\ &= \int_{j\pi}^\infty \frac{\sin^{2k} u}{u} du - \int_{j\pi}^0 \frac{\sin^{2k} v}{v} dv - \int_0^\infty \frac{\sin^{2k} w}{w} dw, \end{aligned}$$

using the respective substitutions $x = u - j\pi$, $x = j\pi - v$, $x = j\pi + w$. Thus

$$\int_0^\infty \left( \frac{\sin^{2k} x}{j\pi + x} + \frac{\sin^{2k} x}{j\pi - x} \right) dx = \int_{j\pi}^\infty \frac{\sin^{2k} x}{x} dx + \int_0^{j\pi} \frac{\sin^{2k} x}{x} dx - \int_0^\infty \frac{\sin^{2k} x}{x} dx = 0$$

as desired.

*II. Solution by Oliver Geupel, Brühl, NRW, Germany.*

We prove the more general statement, that for positive integers $k$ and $m$,

$$\int_0^1 \frac{|\sin x|^m}{(\pi^2 - x^2)((2\pi)^2 - x^2) \cdots ((k\pi)^2 - x^2)} dx = 0.$$

For each positive integer $k$, let $q_k(x) = \prod_{j=1}^k (x - j\pi)(x + j\pi)$. Then

$$\frac{1}{q_k(x)} = \sum_{j=1}^k a_j \left( \frac{1}{x - j\pi} - \frac{1}{x + j\pi} \right)$$

where $a_j = 1/q_k'(jx)$.

We establish, by induction on $k$, the identity

$$\sum_{n=0}^\infty \frac{1}{q_k(x + nx)} = \sum_{n=0}^{k-1} \left( \sum_{j=n+1}^k a_j \right) \left( \frac{1}{x + n\pi} + \frac{1}{x - (n+1)\pi} \right).$$

The case $k = 1$ is covered by the equation

$$\sum_{n=0}^\infty \frac{1}{q_1(x + n\pi)} = a_1 \sum_{n=0}^\infty \left( \frac{1}{x + (n-1)\pi} - \frac{1}{x + (n+1)\pi} \right) = a_1 \left( \frac{1}{x - \pi} + \frac{1}{x} \right).$$

Suppose that the identity has been established when the degree of $q(x)$ is less than $2k$. Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{q_k(x+n\pi)} &= \sum_{n=0}^{\infty}\frac{1}{q_{k-1}(x+n\pi)}+\sum_{n=0}^{\infty}a_k\left(\frac{1}{x+n\pi-k\pi}-\frac{1}{x+nx+k\pi}\right)\\ &= \sum_{n=0}^{k-2}\left(\sum_{j=n+1}^{k-1}a_j\right)\left(\frac{1}{x+n\pi}+\frac{1}{x-(n+1)\pi}\right)+a_k\sum_{j=-k}^{k-1}\frac{1}{x+j\pi}\\ &= \sum_{n=0}^{k-1}\left(\sum_{j=n+1}^{k}a_j\right)\left(\frac{1}{x+n\pi}+\frac{1}{x-(n+1)\pi}\right),\end{aligned}$$

as desired.

For the result, we have that

$$\begin{aligned}\int_0^{\infty}\frac{|\sin x|^m}{q_k(x)}dx &= \sum_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{|\sin x|^m}{q_k(x)}dx=\sum_{n=0}^{\infty}\int_0^{\pi}\frac{|\sin(x+n\pi)|^m}{q_k(x+n\pi)}dx\\ &= \int_0^{\pi}|\sin x|^m\sum_{n=0}^{\infty}\frac{1}{q_k(x+n\pi)}dx=\int_0^{\pi}f(x)dx,\end{aligned}$$

where

$$f(x)=|\sin x|^m\sum_{n=0}^{k-1}\left(\sum_{j=n+1}^{k}a_j\right)\left(\frac{1}{x+n\pi}+\frac{1}{x-(n+1)\pi}\right).$$

Note that $f(x)=-f(\pi-x)$. Therefore,

$$\int_0^{\pi}f(x)dx=-\int_0^{\pi}f(x-\pi)dx=-\int_{\pi}^{0}f(u)(-du)=-\int_0^{\pi}f(x)dx,$$

whence $\int_0^{\pi}f(x)=0$ and the conclusion of the problem follows.

*Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. Perfetti set* $\sin x=\frac{1}{2i}(e^{ix}-e^{-ix})$ *and used contour integration to solve the problem.*

**3708**. [2012 : 24, 26] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Construct the isosceles trapezoid with three equal sides $a$, with a straightedge and a compass alone, provided that its base $b=AB$ and the angle $\alpha$ ($0^\circ<\alpha<90^\circ$) at the base are given.

*Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

Construct rays with initial points $A$ and $B$ that intersect at a point $C$ such that $\angle CAB=\angle CBA=\alpha$. Let the bisector of $\angle CAB$ intersect $BC$ at $D$, and the bisector of $\angle CBA$ intersect $AC$ at $E$. Construct segment $DE$. Then $ABDE$ is the desired isosceles trapezoid. *Proof* : Since $\angle CED=\angle CAB=\alpha$, we have $\angle AED=180^\circ-\alpha$. Since $\angle EAD=\alpha/2$,

$$\angle EDA=180^\circ-\frac{\alpha}{2}-(180^\circ-\alpha)=\frac{\alpha}{2}.$$

It follows that triangle $ADE$ is isosceles with base angles at $A$ and $D$. Thus, $AE = ED$. By symmetry $BD = DE$, so that $AE = ED = DB$.

*Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comǎneşti, Romania; and the proposer. There was one incorrect submission.*

*Most of the constructions used the bisectors of the base angles as in the featured solution. For the solutions of Konečný and of Zvonaru, they constructed the length a after having determined that* $a = \frac{b}{1+2\cos\alpha} = \frac{bt}{t+b}$, *where* $t = AC$ *(in the notation of the featured solution). Bataille located the reference I.M. Yaglom,* Geometric Transformations *I, MAA (1962) pages 132-133, where a more challenging variant of our problem can be found : Given* $\Delta ABC$ *(not necessarily isosceles), construct* $D$ *on side* $BC$ *and* $E$ *on side* $AC$ *such that* $AE = ED = EB$.

**3709**. [2012 : 25, 27] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let $a$, $b$, and $c$ be nonnegative real numbers, $k$ and $l \geq 0$ and define

$$\frac{a+b}{2} - \sqrt{ab} \;=\; k^2\,, \; \frac{a+b+c}{3} - \sqrt[3]{abc} \;=\; l^2\,.$$

Prove that

$$\max(a,b,c) \;\geq\; \min(a,b,c) + \frac{3}{2}(k-l)^2\,.$$

*Solution by Oliver Geupel, Brühl, NRW, Germany, expanded by the editor.*

We set $M = \max(a,b,c)$, $m = \min(a,b,c)$ and assume, without loss of generality, that $a \leq b$.

Note that

$$k^2 = \frac{2}{3}(b-a) - \frac{1}{6}(b + 6\sqrt{ab} - 7a) \leq \frac{2}{3}(M-m)$$

since $0 \leq b - a \leq M$ and $b + 6\sqrt{ab} - 7a \geq 0$. Hence

$$k - \sqrt{\frac{2}{3}(M-m)} \leq 0 \leq l\,. \tag{1}$$

By Schur's inequality we have for $x, y, x \geq 0$, that

$$x^2(x^2-y^2)(x^2-z^2) + y^2(y^2-z^2)(y^2-x^2) + z^2(z^2-x^2)(z^2-y^2) \geq 0$$

which together with the AM-GM inequality then yields

$$x^6 + y^6 + z^6 + 3x^2y^2z^2 \geq \sum_{\text{cyclic}} (x^4y^2 + x^2y^4) \geq 2(x^3y^3 + y^3z^3 + z^3x^3)\,. \tag{2}$$

Setting $x^6 = a$, $y^6 = b$ and $z^6 = c$, then (2) becomes

$$a + b + c + 3\sqrt[3]{abc} \geq 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$$

or

$$\frac{a+b+c}{3} \geq \frac{2}{3}(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) - \sqrt[3]{abc}\,.$$

Hence,

$$l^2 = \frac{a+b+c}{3} - \sqrt[3]{abc} \leq \frac{2}{3}\left(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca}\right) . \tag{3}$$

We now prove the inequality below :

$$2(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca}) \leq 3k^2 + 2(M-m) . \tag{4}$$

To this end we consider three cases separately :

**Case (i).** If $a \leq b \leq c$, then $M - m = c - a$, so (4) becomes

$$2\left(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca}\right) \leq \frac{3(a+b)}{2} - 3\sqrt{ab} + 2(c-a) .$$

The last inequality above, after straightforward computations, reduces to

$$5a + 2\sqrt{ab} + b \leq 4\sqrt{bc} + 4\sqrt{ca}$$

which is true since $4a \leq 4\sqrt{ca}$ and $a + 2\sqrt{ab} + b \leq 4\sqrt{bc}$.

**Case (ii).** If $a \leq c \leq b$, then $M - m = b - a$ and after simplifications (4) reduces to

$$5a + 2\sqrt{ab} + 4c \leq 4\sqrt{ca} + 4\sqrt{bc} + 3b$$

which is true since $5a = 4a+a \leq 4\sqrt{ca}+\sqrt{bc}$, $2\sqrt{ab} \leq 2b$ and $4c = 3c+c \leq 3\sqrt{bc}+b$.

**Case (iii).** If $c \leq a \leq b$, then $M - m = b - c$ and (4) reduces to

$$a + 2\sqrt{ab} + 8c \leq 4\sqrt{bc} + 4\sqrt{ca} + 3b$$

which is true since $a + 2\sqrt{ab} \leq b + 2b = 3b$ and $8c \leq 4\sqrt{bc} + 4\sqrt{ca}$.

This completes the proof of (4).

From (3) and (4) we deduce

$$l^2 \leq k^2 + \frac{2}{3}(M-m) \leq \left(k + \sqrt{\frac{2}{3}(M-m)}\right)^2$$

so

$$l \leq k + \sqrt{\frac{2}{3}(M-m)} . \tag{5}$$

From (1) and (5) we obtain

$$|k - l| \leq \sqrt{\frac{2}{3}(M-m)}$$

and the desired conclusion follows.

*Also solved by the proposer.*

**3710**. [2012 : 23, 25] *Proposed by Billy Jin, Waterloo Collegiate Institute and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Given $n \in \mathbb{N}$, show that there exists a $k \in \mathbb{N}$ such that for all $m \geq k$, there exists a sequence of $m$ consecutive natural numbers which contains exactly $n$ primes.

*Solution by Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA.*

We claim that the statement of the problem holds with $k = p_n$ where $p_n$ represents the $n^{\text{th}}$ prime.

To prove the claim, let $m \geq k$ be an integer, and let $f_m(x)$ be the number of primes in the set $\{x, x+1, \ldots, x+m-1\}$. Then

$$f_m(x+1) = f_m(x) + \epsilon_m(x), \tag{1}$$

where $\epsilon_m(x) \in \{-1, 0, 1\}$. Then, since $m \geq p_n$, we must have $f_m(1) \geq n$ and since the numbers $(m+1)!+2, (m+1)!+3, \ldots, (m+1)!+m+1$ are all composite, we must have $f_m((m+1)!+2) = 0$. Thus, from (1), the list $f_m(1), f_m(2), \ldots, f_m((m+1)!+2)$ contains all integers from 0 to $f_m(1) \geq n$ (and possibly some greater than $f_m(1)$), hence there is a number, $y$, between 1 and $(m+1)!+2$ such that $f_m(y) = n$ and therefore there are exactly $n$ primes in the list $y, y+1, \ldots, y+m-1$.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; RADOUAN BOUKHARFANE, Polytechnique de Montréal, PQ; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposers.*