SKOLIAD No. 143

Lily Yen and Mogens Hansen

In this final Skoliad column we present the solutions to the Mathematics Association of Quebec Contest, Secondary level, 2011, given in Skoliad 137 at [2011:481–483].

1. Otto likes palindromes (numbers that read the same forwards and backwards) so much that he has constructed this alphametic:

$$\text{AMQMA} \times 6 = \text{LUCIE}\,.$$

Find the values of the eight digits.

(Recall that an *alphametic* is a small mathematical puzzle consisting of an equation in which the digits have been replaced by letters. The task is to identify the value of each letter in such a way that the equation comes out true. Different letters have different values, different digits are represented by different letters, and no number begins with a zero. For example, the alphametic PAPA + PAPA = MAMAN has the solution P = 7, A = 5, M = 1, and N = 0, yielding 7575 + 7575 = 15150.)

Solution by Emily Yang, student, Dr. Charles Best Secondary School, Coquitlam, BC.

Since LUCIE only has five digits, AMQMA $< 100\,000 \div 6 \approx 16666.7$, so A = 1. (The leading digit cannot be zero.)

The ones' digit of a product depends only on the ones' digits of the factors, so E = A $\times$ 6 = 6.

If M = 0, then AMQMA $\times$ 6 = 10Q01 $\times$ 6 =???06, so I = 0, but M and I cannot both be zero. Moreover, M $\neq$ 1 since A = 1, and M $\neq$ 6 since E = 6. If M = 2, then AMQMA $\times$ 6 = 12Q21 $\times$ 6 =???26, so I = 2, but M and I cannot both be 2. If M = 4, then AMQMA $\times$ 6 = 14Q41 $\times$ 6 =???46, so I = 4, but M and I cannot both be 4. If M $\geq$ 7, then AMQMA $\times$ 6 $\geq 17\,000 \times 6 = 102\,000$, which has too many digits. This leaves just two possibilities: either M = 3 or M = 5.

If M = 3, then AMQMA $\times$ 6 = 13Q31 $\times$ 6 =???86, so I = 8. If Q = 0, then AMQMA $\times$ 6 = 13031 $\times$ 6 = 78186, but U and I cannot both be 8. If Q = 2, then AMQMA $\times$ 6 = 13231 $\times$ 6 = 79386, but C and M cannot both be 3. If Q $\geq$ 4, then AMQMA $\times$ 6 $\geq$ 13431 $\times$ 6 = 83586, so L = 8, but I and L cannot both be 8. This eliminates all possible values for Q, so M $\neq$ 3.

Thus M = 5. Trying the available values for Q yields: 15051 $\times$ 6 = 90306 (but U and I cannot both be zero), 15251 $\times$ 6 = 91506 (but U and A cannot both be 1), 15351 $\times$ 6 = 92106 (but C and A cannot both be 1), 15451 $\times$ 6 = 92706 (works out), 15751 $\times$ 6 = 94506 (but C and M cannot both be 5), 15851 $\times$ 6 = 95106 (but U and M cannot both be 5), and $15951 \times 6 = 95706$ (but L and Q cannot both be 9).

Thus a single solution exists: $15451 \times 6 = 92706$.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KELLY HO, student, École Banting Middle School, Coquitlam, BC; and DOUGLAS ZHU, student, Meadowridge School, Maple Ridge, BC.

2. Anik is going 108 km/h on the highway. On her way to a math contest, she passes a train that travels beside the highway in the same direction as Anik. She notices that it takes her exactly 77 seconds to pass the train from the rear to the front. Upon arrival, she finds that she has forgotten her calculator and turns back. She again passes the train, which still travels at the same speed. This time it takes her seven seconds to pass from the front of the train to the rear. How long is the train?

Solution by Lucy Yuan, student, New Westminster Secondary School, New Westminster, BC.

Say the train is going at x km/h. Because Anik is traveling in the same direction as the train, her speed relative to the train would be

$$\frac{(108-x)\,\text{km}}{1\,\text{hour}} = \frac{(108-x)\,\text{km}}{3600\,\text{seconds}}\,.$$

Now, when traveling in the same direction as the train, it takes Anik 77 seconds to pass the train, so if ℓ is the length (in km) of the train, then

$$\frac{108-x}{3600}\cdot 77 = \ell\,.$$

When Anik passes the train in the opposite direction, her speed relative to the train is

$$\frac{(108+x)\,\text{km}}{1\,\text{hour}} = \frac{(108+x)\,\text{km}}{3600\,\text{seconds}}\,.$$

At this speed it takes Anik 7 seconds to pass, so

$$\frac{108+x}{3600}\cdot 7 = \ell\,.$$

Thus

$$\frac{108-x}{3600}\cdot 77 = \frac{108+x}{3600}\cdot 7\,,$$

so, solving for x we get

$$\begin{aligned}(108-x)\cdot 77 &= (108+x)\cdot 7\\ 11(108-x) &= 108+x\\ 1188-11x &= 108+x\\ 1080 &= 12x\\ x &= 90.\end{aligned}$$

Copyright © Canadian Mathematical Society, 2014

Therefore

$$\ell = \frac{108+x}{3600}\cdot 7 = \frac{108+90}{3600}\cdot 7 = 0.385,$$

so the length of the train is 0.385 km or 385 m.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and SAMUEL HUANG, student, Moscrop Secondary School, Burnaby, BC.

3. At an intersection, the traffic light is red for 30 seconds and green for 30 seconds. (Ignore the yellow light.) How long do you have to wait, on average, at the intersection? Justify your answer.

Solution by Douglas Zhu, student, Meadowridge School, Maple Ridge, BC.

If the traffic light is green when you arrive at the intersection, the (average) wait is obviously 0 seconds.

If the traffic light is red when you arrive, you may have to wait t seconds or $30-t$ seconds ($0 \le t \le 15$), and these two possibilities are equally likely (for each value of t). Thus, for each value of t, the average wait is $\frac{t+(30-t)}{2} = \frac{30}{2} = 15$ seconds. Since this average is independent of t, the average wait, if the light is red upon arrival, is 15 seconds.

When you arrive at the intersection, the probability that the light is green is $\frac{1}{2}$ and the probability that the light is red is $\frac{1}{2}$. Therefore the average wait is $0\cdot\frac{1}{2} + 15\cdot\frac{1}{2} = \frac{15}{2} = 7.5$ seconds.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

That you must wait an average of 15 *seconds if you arrive at a red light is either intuitively clear or somewhat technical to prove. The problem is that the probability of waiting* exactly t *seconds is infinitely small, regards of the value of* t. *Since* t *can take infinitely many values, you are now adding infinitely many infinitely small values. To accomplish such a feat is the task of calculus, which is beyond Skoliad.*

4. How many integers from 0 to 999 (inclusive) do not contain the digit 7? What is the sum of these numbers?

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

If you allow leading zeroes, the task is to count the number of three-digit numbers that do not use the digit 7. Each digit has nine possibilities: 0, 1, 2, 3, 4, 5, 6, 8, and 9. Therefore $9^3 = 729$ such numbers exist.

Consider all three-digit numbers with ones' digit 5. They have the form $ab5$, where a and b are (possibly identical) digits other than 7. Thus there are nine possibilities for a and nine for b, so $9^2 = 81$ numbers (without 7) have ones' digit 5.

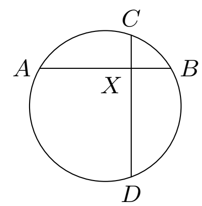
This is of course true for any digit, not just for 5, and it is true for the tens' digit and the hundreds' digit as well: the digit d occurs as ones' digit 81 times, as tens' digit 81 times, and as hundreds' digit 81 times. Therefore the sum of all such numbers is

$$81 \cdot 111 \cdot (0+1+2+3+4+5+6+8+9) = 81 \cdot 111 \cdot 38 = 341\,658\,.$$

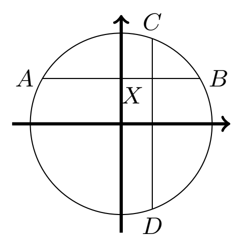
Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

5. In a circle with radius r, the two chords AB and CD intersect at a right angle at X. Show that

$$|XA|^2 + |XB|^2 + |XC|^2 + |XD|^2 = 4r^2\,.$$



Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.



Impose a coordinate system such that $(0,0)$ is the centre of the circle and the chords are parallel to the axes. Let (x,y) be the coordinates of X, and let (a,y) be the coordinates of A. Since A is on the circle, $a^2+y^2 = r^2$, so $a = -\sqrt{r^2-y^2}$. Hence $|XA| = x - a = x + \sqrt{r^2-y^2}$.

Similarly,

$$\begin{aligned} |XB| &= \sqrt{r^2-y^2} - x, \\ |XC| &= \sqrt{r^2-x^2} - y, \text{ and} \\ |XD| &= y + \sqrt{r^2-x^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} |XA|^2 &= \left(x + \sqrt{r^2-y^2}\right)^2 = x^2 + 2x\sqrt{r^2-y^2} + \left(\sqrt{r^2-y^2}\right)^2 \\ &= x^2 + 2x\sqrt{r^2-y^2} + r^2 - y^2, \end{aligned}$$

and similarly

$$\begin{aligned} |XB|^2 &= r^2 - y^2 - 2x\sqrt{r^2-y^2} + x^2, \\ |XC|^2 &= r^2 - x^2 - 2y\sqrt{r^2-x^2} + y^2, \text{ and} \\ |XD|^2 &= y^2 + 2y\sqrt{r^2-x^2} + r^2 - x^2. \end{aligned}$$

Thus

$$|XA|^2 + |XB|^2 = 2x^2 - 2y^2 + 2r^2$$

and

$$|XC|^2 + |XD|^2 = 2r^2 - 2x^2 + 2y^2,$$

so

$$|XA|^2 + |XB|^2 + |XC|^2 + |XD|^2 = 4r^2$$

as required.

Copyright © Canadian Mathematical Society, 2014

6. The cubes $173^3 = 5\,177\,717$, $192^3 = 7\,077\,888$ and $1309^3 = 2\,242\,946\,629$ are examples of a whole number N that contains as many different digits as its cube, N^3. If N^3 contains fewer different digits than N, then N is said to be *deficient*. For example, 13 798 has five different digits, while its cube, 2 626 929 525 592, has four (2, 5, 6, and 9), so 13 798 is deficient. Show that there are infinitely many deficient whole numbers.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

If $N = 1\,023\,456\,789$, then

$$N^3 = 1\,072\,033\,936\,267\,303\,561\,560\,897\,069\,.$$

Since N contains all ten digits while N^3 does not contain the digit 4, N is deficient.

If k is a positive integer, then $N \cdot 10^k$ clearly still contains all ten digits, while $(N \cdot 10^k)^3 = N^3 \cdot 10^{3k}$ consists of the same digits as N^3 followed by $3k$ zeroes. Therefore $(N \cdot 10^k)^3$ does not contain the digit 4, so $(N \cdot 10^k)^3$ is deficient for every positive integer k. Thus infinitely many deficient integers exist.

Our solver must have used a tool more powerful than a pocket calculator to find the cube of his ten-digit number N. A more modest example will do. Indeed, the problem states that $192^3 = 7\,077\,888$ from which follows that $1920^3 = 7\,077\,888\,000$. Since 1920 contains four different digits while 1920^3 only contains three, 1920 is deficient. Our solver's idea of looking at $1920 \cdot 10^k$ now yields an infinitude of deficient integers.

7. If x, y, and z are real numbers such that

$$x = \sqrt{11 - 2yz}, \quad y = \sqrt{12 - 2xz}, \quad \text{and} \quad z = \sqrt{13 - 2xy}$$

what is the value of $x + y + z$?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Note that

$$\begin{aligned}(x+y+z)^2 &= (x+y+z)(x+y+z)\\ &= x^2 + xy + xz + yx + y^2 + yz + zx + zy + z^2\\ &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.\end{aligned}$$

Now, $x^2 = 11 - 2yz$, $y^2 = 12 - 2xz$, and $z^2 = 13 - 2xy$. Therefore,

$$\begin{aligned}(x+y+z)^2 &= 11 - 2yz + 12 - 2xz + 13 - 2xy + 2xy + 2xz + 2yz\\ &= 36,\end{aligned}$$

so $x + y + z = \pm 6$. Since x, y, and z all equal square roots, they are nonnegative, so their sum cannot be negative. Thus $x + y + z = 6$.

This issue's prize for the best solutions goes to Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany. We thank our readers for the solutions it has been our privilege to receive, edit, and publish during our tenure as Skoliad editors.