PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Electronic submissions are preferable, with each solution contained in a separate file. Solution files should be named using the convention LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions to the editor at* `crux-editors@cms.math.ca`*. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Original problems are particularly sought, but other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 May 2014***, although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3783. *Correction. Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$(3a^2+2)\frac{a^3+b^3}{a^2+ab+b^2} + (3b^2+2)\frac{b^3+c^3}{b^2+bc+c^2} + (3c^2+2)\frac{c^3+a^3}{c^2+ca+a^2} \ge 10abc\,.$$

3792. *Correction. Proposed by Marcel Chiriţă, Bucharest, Romania.*

Solve the following system

$$2^x + 2^y = 12$$
$$3^x + 3^y = 36$$

for $x, y \in \mathbb{R}$.

3801. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Triangle ABC is isosceles with $AB = AC$ and $\angle A = 100^\circ$. Let D be the point on AB such that $\angle BCD = 10^\circ$ and let E be the point on BC such that $EC = AC$. Determine the point K on CD such that triangles KAD and KCE have equal areas.

3802. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Solve the following system

$$\sqrt{2x+1} + \sqrt{3y+1} + \sqrt{4z+1} = 15$$
$$3^{2x+\sqrt{3y+1}} + 3^{3y+\sqrt{4z+1}} + 3^{4z+\sqrt{2x+1}} = 3^{30}$$

for $x, y, z \in \mathbb{R}$.

3803. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{a^2+ca} + \sqrt{b^2+ab} + \sqrt{c^2+bc} \le \sqrt{2}(a+b+c)\,.$$

3804. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Let $ABCD$ be a convex quadrilateral. Construct, using only compass and straight-edge, the line parallel to one side of the quadrilateral which bisects its area.

3805. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC be a triangle with incentre I. Let A' be on ray IA beyond A such that $A'A = BC$. Let B' and C' be similarly defined, such that $B'B = CA$ and $C'C = AB$. Prove that

$$\frac{[A'B'C']}{[ABC]} \ge (1+\sqrt{3})^2,$$

where $[\cdot]$ denotes the area.

3806. *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC with angles $\alpha, \beta, \gamma \ne 90^\circ$ be inscribed in a circle with centre O and radius R, and let U, V, W be the centres of the hyperbolas with parameter R, focus O and associated directrices BC, CA, AB, respectively. Prove that

$$[UVW] \times [ABC] = R^4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

where $[\cdot]$ denotes the area.

3807. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let ABC be a triangle with incentre I through which an arbitrary line passes meeting sides AB and AC at the points D and E respectively. Show that

$$\frac{1}{r} \geq \frac{1}{AD} + \frac{1}{AE}$$

where r denotes the inradius of ABC.

3808. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC be a triangle with sides a, b, c, angle bisectors AT_a, BT_b, CT_c and medians $m_a = AM_a, m_b = BM_b, m_c = CM_c$. These lines define a new triangle with vertices

$$A' = BT_b \cap CM_c \quad B' = CT_c \cap AM_a \quad \text{and} \quad C' = AT_a \cap BM_b,$$

with angles α at A', β at B', and γ at C'. Prove that

$$\frac{m_a m_b m_c \sin\alpha \sin\beta \sin\gamma}{(a+2b)(b+2c)(c+2a)} = \frac{r}{32R},$$

where R is the circumradius and r is the inradius of ABC.

3809. *Proposed by Michel Bataille, Rouen, France.*

For positive real numbers x, y, let

$$G(x,y) = \sqrt{xy}, \quad A(x,y) = \frac{x+y}{2}, \quad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}\,.$$

Prove that

$$G(x^x, y^y) \geq (Q(x,y))^{A(x,y)}\,.$$

3810. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k > 0$ be a positive real number. Find the value of

$$\int_0^1 \int_0^1 \left\{ \frac{x^k}{y} \right\} dx dy,$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a.

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3783. *Correction. Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b, c trois nombres réels positifs avec $ab + bc + ca = 3$. Montrer que

$$(3a^2+2)\frac{a^3+b^3}{a^2+ab+b^2} + (3b^2+2)\frac{b^3+c^3}{b^2+bc+c^2} + (3c^2+2)\frac{c^3+a^3}{c^2+ca+a^2} \geq 10abc\,.$$

3792. *Correction. Proposé par Marcel Chiriţă, Bucarest, Roumanie.*

Résoudre le système suivant

$$\begin{aligned} 2^x + 2^y &= 12 \\ 3^x + 3^y &= 36 \end{aligned}$$

for $x, y \in \mathbb{R}$.

3801. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit ABC un triangle isocèle avec $AB = AC$ et $\angle A = 100°$. Soit D le point sur AB tel que $\angle BCD = 10°$ et soit E le point sur BC tel que $EC = AC$. Déterminer le point K sur CD tel que les aires des triangles KAD et KCE soient égales.

3802. *Proposé par Marcel Chiriţă, Bucarest, Roumanie.*

Résoudre le système suivant

$$\begin{aligned} \sqrt{2x+1} + \sqrt{3y+1} + \sqrt{4z+1} &= 15 \\ 3^{2x+\sqrt{3y+1}} + 3^{3y+\sqrt{4z+1}} + 3^{4z+\sqrt{2x+1}} &= 3^{30} \end{aligned}$$

pour $x, y, z \in \mathbb{R}$.

3803. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\sqrt{a^2 + ca} + \sqrt{b^2 + ab} + \sqrt{c^2 + bc} \le \sqrt{2}(a + b + c)\,.$$

3804. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

Soit $ABCD$ un quadrilatère convexe. En utilisant uniquement la règle et le compas, construire la droite parallèle à un côté du quadrilatère qui divise son aire en deux moitiés.

3805. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit ABC un triangle et I son centre de gravité. Soit A' sur le rayon IA au-delà de A de sorte que $A'A = BC$. Soit B' et C' définis de manière analogue, de sorte que $B'B = CA$ et $C'C = AB$. Montrer que

$$\frac{[A'B'C']}{[ABC]} \ge (1 + \sqrt{3})^2,$$

où $[\cdot]$ désigne la surface.

3806. *Proposé par Michel Bataille, Rouen, France.*

Soit un triangle ABC, d'angles $\alpha, \beta, \gamma \neq 90^\circ$, inscrit dans un cercle de centre O et de rayon R, et soit respectivement U, V, W les centres des hyperboles de paramètre R, de foyer O et de directrices associées BC, CA, AB. Montrer que

$$[UVW] \times [ABC] = R^4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

où $[\cdot]$ désigne la surface.

3807. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit ABC un triangle avec I comme centre du cercle inscrit par lequel passe une droite arbitraire coupant respectivement les côtés AB et AC aux points D et E. Montrer que

$$\frac{1}{r} \geq \frac{1}{AD} + \frac{1}{AE}$$

où r dénote le rayon du cercle inscrit de ABC.

3808. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit ABC un triangle de côtés a, b, c, de bissectrices AT_a, BT_b, CT_c, et de médianes $m_a = AM_a, m_b = BM_b, m_c = CM_c$. Ces droites définissent un nouveau triangle de sommets

$$A' = BT_b \cap CM_c \quad B' = CT_c \cap AM_a \quad \text{et} \quad C' = AT_a \cap BM_b,$$

d'angles α en A', β en B' et γ en C'. Montrer que

$$\frac{m_a m_b m_c \sin\alpha \sin\beta \sin\gamma}{(a+2b)(b+2c)(c+2a)} = \frac{r}{32R},$$

où R est le rayon du cercle circonscrit de ABC et r celui de son cercle inscrit.

3809. *Proposé par Michel Bataille, Rouen, France.*

Soit

$$G(x,y) = \sqrt{xy}, \quad A(x,y) = \frac{x+y}{2}, \quad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}$$

où x et y sont des nombres réels positifs. Montrer que

$$G(x^x, y^y) \geq (Q(x,y))^{A(x,y)}.$$

3810. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $k > 0$ un nombre réel positif. Trouver la valeur de

$$\int_0^1 \int_0^1 \left\{ \frac{x^k}{y} \right\} dxdy,$$

où $\{a\} = a - \lfloor a \rfloor$ désigne la partie fractionnaire de a.