

PROBLEM OF THE MONTH

No. 4

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*This column is dedicated to the memory of former **CRUX with MAYHEM** Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of **CRUX with MAYHEM**, and, through his work on mathematics contests and outreach programs, with many others. The “Problem of the Month” features a problem and solution that we know Jim would have liked.*



This problem comes from the 2004 Canadian Open Mathematics Challenge, a contest for high school students.

In a sumac sequence, t_1, t_2, \dots, t_m , each term is an integer greater than or equal to 0. Also, each term, starting with the third, is the difference between the preceding two terms:

$$t_{n+2} = t_n - t_{n+1}, n \geq 1.$$

The sequence terminates at t_m if t_m is greater than t_{m-1} . For example,

$$120, 71, 49, 22, 27$$

is a sumac sequence of length 5.

- (a) *In the set of sumac sequences that have $t_1 = 150$, find the positive integer B such that the sumac sequence $150, B, \dots$, has maximum length.*
- (b) *Let m be a positive integer greater than or equal to 5. Determine the number of sumac sequences of length m in which $t_m \leq 2000$ and in which no term is divisible by 5.*

Part (a) of this problem appeared on the 1980 Michigan Mathematics Prize Competition [1984 : 42; 1985 : 10-12] and its solution is given in my *More Mathematical Morsels* (Dolciani Series, Vol. 10, MAA, 1991, pages 191-194). The outstanding problem poser and solver Murray Klamkin (University of Alberta) found that, for arbitrary first term A , the value of B that gives the sequence of maximum length is either $[\tau A]$ or $[\tau A] + 1$, where $\tau = \frac{1 + \sqrt{5}}{2}$, is the golden ratio, and the square brackets indicate the integer part of its argument. However, our interest here is in part (b).

The key to the part (b) is the observation that a sumac sequence is a general Fibonacci sequence in disguise: 120, 71, 49, 22, 27 is just the Fibonacci sequence 27, 22, 49, 71, 120 written backwards.

In order for a sumac sequence to terminate, its last term x must exceed its second-last term y :

$$t_1, t_2, \dots, y, x,$$

where $x > y$. A Fibonacci sequence that begins x, y, \dots , where $x > y$,

$$x, y, x + y, x + 2y, 2x + 3y, 3x + 5y, \dots,$$

being of infinite length, can be truncated and reversed at any term to yield a sumac sequence of every length $m \geq 2$. It appears, then, that our task is to count the Fibonacci sequences that

- (i) begin x, y ,
- (ii) have $x > y$,
- (iii) have $x \leq 2000$, and
- (iv) have no term divisible by 5.

Since each such Fibonacci sequence provides a sumac sequence of every length, the number of sumac sequences of length m that are generated by these Fibonacci sequences is the same for every $m \geq 2$. It is easy to find additional Fibonacci sequences which yield acceptable sumac sequences of lengths 2, 3, and 4. For example, reversing $17, 2, 19, 21, 40, \dots$, yields the sumac sequence 2, 17 of length 2, the sequence 19, 2, 17 of length 3, and 21, 19, 2, 17 of length 4, none of which has a term that is divisible by 5. The sequence 40, 21, 19, 2, 17, of length 5, is not acceptable since 40 is divisible by 5. As we shall see, however, the acceptable sumac sequences of length $m \geq 5$ are generated *exclusively* by the Fibonacci sequences we are proposing to count. Mercifully, then, the number of acceptable sumac sequences of length $m \geq 5$ is the same for every m .

Clearly no term of such a Fibonacci sequence can end in 0 or 5. Now, the last digits of x and y set the last digits of the entire sequence. For example, if x ends in 7 and y in 2, then the last digits of the Fibonacci sequence are 7, 2, 9, 1, 0, 1, 1, 2, \dots . The presence of the 0 implies that every such sequence of length at least 5 contains a multiple of 5 and therefore, when reversed, fails to yield an acceptable sumac sequence of any length $m \geq 5$. Thus the pair (7, 2) is not an acceptable pair for the final digits (a, b) of x and y .

Let us proceed to determine the acceptable pairs (a, b) . As noted above, neither a nor b can itself be 0 or 5. Now, we mustn't assume, just because x is greater than y , that the last digit of x must be bigger than the last digit of y . I am embarrassed to confess that I thoughtlessly fell into this trap the first time I tried the problem and consequently failed to count almost half the sequences. There is nothing for it but to check all 64 of the *ordered* pairs (a, b) where $a, b \in \{1, 2, 3, 4, 6, 7, 8, 9\}$.

This threatens to be a long, tedious undertaking, but our salvation lies in the fact that if the n^{th} term f_n of a Fibonacci sequence is divisible by 5, then so

is the term f_{n-5} :

$$f_n = f_{n-1} + f_{n-2} = 2f_{n-2} + f_{n-3} = 3f_{n-3} + 2f_{n-4} = 5f_{n-4} + 3f_{n-5};$$

since 3 and 5 are relatively prime, 5 divides f_n implies 5 divides f_{n-5} .

Hence, if any term f_n is divisible by 5, it follows that, proceeding toward the beginning of the sequence in steps of 5 terms, one of the first five terms will also be divisible by 5. Therefore we need only check the first five terms of a sequence of last digits in order to discover whether a pair (a, b) is acceptable.

At this point, it is evident that there is a one-to-one correspondence between the Fibonacci sequences that we are in the process of counting and the acceptable sumac sequences, *in view of their length being at least 5*, thus confirming our earlier claim that the acceptable sumac sequences are generated exclusively by our Fibonacci sequences:

since no term of a sumac sequence is divisible by 5, in particular none of its last 5 terms (which are guaranteed by $m \geq 5$), none of the first 5 terms of the corresponding Fibonacci sequence is divisible by 5; therefore no term whatsoever of the Fibonacci sequence is divisible by 5, and consequently the Fibonacci sequence generates an acceptable sumac sequence of every length $m \geq 5$.

Not needing to check more than the first 5 terms of a sequence of last digits, it doesn't take very long to find that there are 16 acceptable ordered pairs (a, b) :

$$(9, 7), (9, 2), (8, 9), (3, 9), (8, 4), (6, 8), (1, 8), (7, 6), \\ (7, 1), (4, 7), (6, 3), (2, 6), (4, 2), (3, 4), (1, 3), (2, 1).$$

Consider the pair $(9, 7)$. The largest $x \leq 2000$ that ends in 9 is 1999 and it occurs in each of the 200 $(9, 7)$ -pairs:

$$(1999, 1997), (1999, 1987), (1999, 1977), \dots, (1999, 7)$$

Each of these pairs gives a Fibonacci sequence that begins with 1999, and hence a sumac sequence that ends in 1999, of every length at least 5 (in fact, greater than or equal to 2). For example, the pair $(1999, 1997)$ yields the Fibonacci sequence

$$1999, 1997, 3996, 5993, 9989, \dots$$

and hence the sumac sequence of length 5

$$9989, 5993, 3996, 1997, 1999.$$

Thus the largest x , 1999, gives rise to a total of 200 sumac sequences of length at least 5.

The next largest value of x is 1989 and it occurs in the 199 $(9, 7)$ -pairs

$$(1989, 1987), (1989, 1977), \dots, (1989, 7).$$

Similarly for $x = 1979, 1969, \dots, 19, 9$, which occur, respectively, in 198, 197, \dots , 2, 1 pairs. Altogether, then, the pair $(9, 7)$ gives rise to a grand total of

$$1 + 2 + \dots + 200 = \frac{200(201)}{2} = 20\,100$$

sequences.

Similarly, whenever $a > b$, the pair (a, b) yields 20 100 sequences. However, when $a < b$, there are slightly fewer sequences.

Consider the pair $(4, 7)$. The greatest x is 1994, and it pairs with each smaller positive integer that ends in 7. Since 4 is less than 7, these smaller integers start at 1987, giving 199 pairs. Again, $x = 1984$ pairs with 198 smaller positive integers ending in 7, starting at 1977, and so on for a total of

$$199 + 198 + \dots + 1 + 0 = 19\,900$$

(for $x = 4$ there are 0 smaller positive integers that end in 7).

Since there are eight ordered pairs (a, b) with $a > b$ and eight with $a < b$, the grand total of acceptable sequences is

$$8(20\,100) + 8(19\,900) = 8(40\,000) = 320\,000.$$

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