

THE OLYMPIAD CORNER

No. 309

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName.FirstName.OCProblemNumber (example Doe.Jane.OC1234.tex). It is preferred that readers submit a \LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at crux-olympiad@cms.math.ca. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page.

To facilitate their consideration, solutions to the problems should be received by the editor by **1 May 2014**, although solutions received after this date will also be considered until the time when a solution is published.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet, of l'Université Saint-Boniface in Winnipeg, for translations of the problems.

OC111. Let x, y and z be positive real numbers. Show that

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4.$$

OC112. Find all pairs of natural numbers (a, b) such that

$$\gcd(a, b) + 9\text{lcm}(a, b) + 9(a + b) = 7ab.$$

OC113. Prove that among any n vertices of a regular $(2n - 1)$ -gon, where $n \geq 3$, we can find 3 which form an isosceles triangle.

OC114. Let ABC be a scalene triangle. Its incircle touches BC, AC, AB at D, E, F respectively. Let L, M, N be the symmetric points of D, E, F with respect to EF, FD , and DE , respectively. The line AL intersects BC at P , the line BM intersects CA at Q , and the line CN intersects AB at R . Prove that P, Q, R are collinear.

OC115. Find the smallest positive integer n for which there exists a positive integer k such that the last 2012 decimal digits of n^k are all 1's.

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OC111. Soit x, y et z trois nombres réels positifs. Démontrez que

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4.$$

OC112. Déterminer toutes les paires de nombres naturels (a, b) tels que

$$\text{pgcd}(a, b) + 9 \text{ppcm}(a, b) + 9(a + b) = 7ab.$$

OC113. Démontrer que parmi n'importe quels n sommets d'un polygone régulier à $(2n - 1)$ sommets, où $n \geq 3$, on peut en tirer 3 qui forment un triangle isocèle.

OC114. Soit ABC un triangle scalène. Son cercle inscrit touche BC , AC et AB aux points D , E et F respectivement. Soient L , M et N les points symétriques à D , E et F par rapport à EF , FD et DE respectivement. La ligne AL intersecte BC en P , la ligne BM intersecte CA en Q et la ligne CN intersecte AB en R . Démontrer que P , Q et R sont colinéaires.

OC115. Déterminer le plus petit entier positif n pour lequel il existe un entier positif k tel que les 2012 dernières positions décimales de n^k sont toutes 1.

OLYMPIAD SOLUTIONS

OC51. Determine all pairs (a, b) of nonnegative integers so that $a^b + b$ divides $a^{2b} + 2b$. Note, for this problem $0^0 = 1$.
(Originally question 4 from the second day of Austrian Mathematical Olympiad.)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Comănești, Romania. We give the solution of Wang.

The only solutions are $(a, 0)$, $(0, b)$ for $a, b \geq 0$ and $(a, b) = (2, 1)$. Since $(a, 0)$ and $(0, b)$ clearly satisfy the equation, it remains to show that if $a \geq 1$ and $b \geq 1$ then $(a, b) = (2, 1)$.

Suppose that (a, b) is a solution where $a, b \geq 1$. If $a = 1$ then the condition becomes $1 + b \mid 1 + 2b$ which is impossible since $1 + b < 1 + 2b < 2(1 + b)$. Hence we have $a \geq 2$.

As $a^b + b \mid a^{2b} + 2b$ we have

$$\begin{aligned} a^b + b &= \gcd(a^b + b, a^{2b} + 2b) = \gcd(a^b + b, a^b(a^b + b) - (a^{2b} + 2b)) \\ &= \gcd(a^b + b, ba^b - 2b) = \gcd(a^b + b, b(a^b + b) - (ba^b - 2b)) \\ &= \gcd(a^b + b, b^2 + 2b) \end{aligned}$$

Thus

$$a^b + b \leq b^2 + 2b.$$

Hence

$$a^b \leq b^2 + b. \quad (1)$$

We claim that (1) cannot hold if $a \geq 3$ by proving by induction that

$$3^b > b^2 + b$$

for all $b \geq 1$. This is clear when $b = 1$.

Suppose that $3^b > b^2 + b$ for some $b \geq 1$. Then

$$\begin{aligned} 3^{b+1} &> 3(b^2 + b) = b^2 + 2b + b^2 + b + b^2 \\ &\geq b^2 + 2b + 1 + b + 1 = (b + 1)^2 + (b + 1), \end{aligned}$$

completing the induction.

It remains to consider the case when $a = 2$. Using the same induction argument, it can be proved easily that

$$2^b > b^2 + b$$

for all $b \geq 5$, which contradicts (1). Thus we only need to check the cases $(a, b) \in \{(2, 1), (2, 2), (2, 3), (2, 4)\}$, and a direct computation shows that only $(2, 1)$ yields a solution.

This completes the proof.

OC52. Let d, d' be two divisors of n with $d' > d$. Prove that

$$d' > d + \frac{d^2}{n}.$$

(Originally question 1 from the Russia National Olympiad 2011: Grade 11.)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Marian Dincă, Bucharest, Romania; Oliver Geupel, Brühl, NRW, Germany; Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA; Kim Uyen Truong, California State University, Fullerton, CA, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Truong.

Since d and d' are divisors of n there exist some a, b so that

$$ad' = bd = n.$$

As $d' > d$ we have $a < b$ and hence $a \leq b - 1$. Thus

$$\begin{aligned} d' > d + \frac{d^2}{n} &\Leftrightarrow \frac{n}{a} > \frac{n}{b} + \frac{n}{b^2} \\ &\Leftrightarrow \frac{1}{a} > \frac{1}{b} + \frac{1}{b^2} \\ &\Leftrightarrow b^2 > ab + a = a(b + 1). \end{aligned}$$

This last inequality holds as

$$a(b + 1) \leq (b - 1)(b + 1) = b^2 - 1 < b^2.$$

Thus,

$$d' > d + \frac{d^2}{n}.$$

OC53. Find all the polynomials $P(x) \in \mathbb{R}[x]$ so that $P(a) \in \mathbb{Z}$ implies $a \in \mathbb{Z}$. (Originally question 4 from the 2011 Singapore National Olympiad.)

Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany and Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA. We give the solution of Curtis.

Case 1: $\deg(P) = 0$. Then, there exists a constant c so that $P(x) = c$ for all real x . Then this polynomial has the desired property if and only if c is not an integer. [If c is not an integer, then P has the desired property by vacuity].

Case 2: $\deg(P) = 1$. Then $P(x) = ax + b$ for some real numbers a ($\neq 0$) and b . As the range of P is the real numbers, for each integer m there is an integer x_m so that $P(x_m) = m$. Then

$$ax_m + b = m \quad \text{and} \quad ax_{m-1} + b = m - 1,$$

which implies that

$$x_{m-1} = \frac{(m-1) - b}{a} \quad \text{and} \quad x_m = \frac{m - b}{a}.$$

Subtracting, we get that

$$\frac{1}{a} = x_m - x_{m-1} \in \mathbb{Z}.$$

Thus $a = \frac{1}{k}$ with k an integer. Then

$$x_m = \frac{m - b}{a} = km - kb \Rightarrow kb = x_m - km \in \mathbb{Z}.$$

Let $l = kb \in \mathbb{Z}$, then we get

$$P(x) = \frac{x+l}{k}.$$

It is easy to see that this polynomial has the desired property.

Case 3: $\deg(P) \geq 2$. We claim there is no such polynomial in this case.

Since for any polynomial P which satisfies the condition, $-P$ also satisfies the condition, without loss of generality we can assume that the leading coefficient of P is positive. We then have $\lim_{x \rightarrow \infty} P'(x) = \infty$, and so there exists some x_0 so that $P'(x) \geq 2$ for all $x \geq x_0$.

Moreover, as $\lim_{x \rightarrow \infty} P(x) = \infty$, by the Intermediate Value Theorem there exists some M so that the interval

$$[M, \infty) \subset P((x_0, \infty)).$$

Thus, for every $m > M$ there exist two distinct integers $u, v > x_0$ so that

$$P(u) = m; P(v) = m + 1.$$

By the Mean Value Theorem, there exists some c_m between u, v so that

$$P'(c_m) = \frac{P(u) - P(v)}{u - v} = \frac{1}{u - v} < 1.$$

But as $c_m > x_0$ we also have $P'(c_m) \geq 2$, a contradiction. Thus there is no polynomial of degree 2 or higher satisfying this condition.

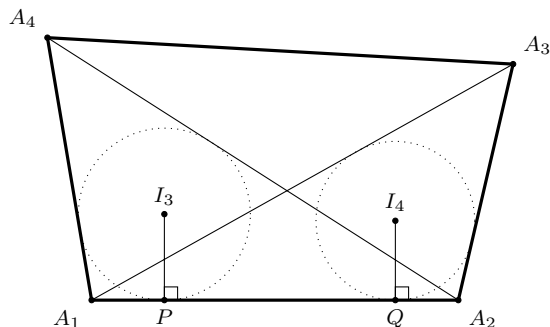
In summary, a polynomial P satisfies this condition if and only if either P is a nonintegral constant or $P(x) = \frac{x+l}{k}$ for some integers l, k with $k \neq 0$.

OC54. Given four points in the plane so that the incircles of the four triangles formed by three of the four points are equal, prove that the four triangles are equal. (*Originally question 5 from the 2011 Japan National Olympiad.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We show that the four points are the vertices of a rectangle.

Let the four points be A_1, A_2, A_3 , and A_4 . Let a_{ik} denote the length of the segment A_iA_k . Let I_1, I_2, I_3 , and I_4 denote the incentres of the triangles $A_2A_3A_4, A_3A_4A_1, A_4A_1A_2$, and $A_1A_2A_3$, respectively. If the convex hull \mathcal{C} of the four points is a triangle, say, triangle $A_1A_2A_3$, then A_4 is an interior point of that triangle, which implies that the inradius of triangle $A_1A_2A_3$ is greater than the inradius of triangle $A_1A_2A_4$, a contradiction. Hence, \mathcal{C} is a quadrilateral, say, the convex quadrilateral $A_1A_2A_3A_4$. Let P and Q be the orthogonal projections of I_3 and I_4 on the line A_1A_2 . Because triangles $A_1A_2A_4$ and $A_1A_2A_3$ have congruent incircles, the quadrilateral PQI_4I_3 is a rectangle.



We obtain

$$\begin{aligned} I_3I_4 &= PQ = A_1A_2 - A_1P - A_2Q \\ &= a_{12} - \frac{a_{12} + a_{14} - a_{24}}{2} - \frac{a_{12} + a_{23} - a_{13}}{2} \\ &= \frac{a_{13} + a_{24} - a_{14} - a_{23}}{2}. \end{aligned}$$

Similarly,

$$I_1I_2 = \frac{a_{13} + a_{24} - a_{14} - a_{23}}{2},$$

thus $I_1I_2 = I_3I_4$. Analogously, $I_4I_1 = I_2I_3$. Hence, the quadrilateral $I_1I_2I_3I_4$ is a parallelogram.

We obtain $A_1A_2 \parallel I_3I_4 \parallel I_1I_2 \parallel A_3A_4$ and similarly $A_1A_4 \parallel A_2A_3$. Thus, the quadrilateral $A_1A_2A_3A_4$ is a parallelogram. Therefore,

$$\begin{aligned} a_{12} + a_{14} + a_{24} &= \frac{2[A_1A_2A_4]}{I_3P} = \frac{2[A_1A_2A_3]}{I_4Q} \\ &= a_{12} + a_{23} + a_{13} = a_{12} + a_{14} + a_{13}. \end{aligned}$$

Consequently, $A_2A_4 = a_{24} = a_{13} = A_1A_3$.

This shows that the quadrilateral $A_1A_2A_3A_4$ is a rectangle.

OC55. Let d be a positive integer. Show that for every integer S there exists an integer $n > 0$ and a sequence $\epsilon_1, \epsilon_2, \dots, \epsilon_n$, where for any k , $\epsilon_k = 1$ or $\epsilon_k = -1$, such that

$$S = \epsilon_1(1 + d)^2 + \epsilon_2(1 + 2d)^2 + \epsilon_3(1 + 3d)^2 + \dots + \epsilon_n(1 + nd)^2.$$

(Originally question 5 from 2011 Canadian Mathematical Olympiad.)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

Let S be any integer. We are going to show that S can be written in the required form.

Let m be an integer such that $m \geq 4d^3$ and that the number

$$S_0 = - \sum_{k=1}^m (1 + kd)^2,$$

has the same parity as S . Define now

$$S_k = S_{k-1} + 2(1 + 2kd^2)^2$$

for $k = 1, 2, 3, \dots, 2d^2 - 1$. We have

$$S_k \equiv S_{k-1} + 2 \pmod{4d^2}.$$

Hence, the sequence

$$S_0, S_1, S_2, \dots, S_{2d^2-1}$$

includes a number that has the same remainder modulo $4d^2$ as S . Let S' be that number.

Then, S' can be written in the required form and there is an integer q such that

$$S = S' + 4d^2q.$$

Observe that for each integer k we have

$$(1 + kd)^2 - (1 + (k + 1)d)^2 - (1 + (k + 2)d)^2 + (1 + (k + 3)d)^2 = 4d^2.$$

Therefore, for each $q \geq 0$ we have

$$4d^2q = \sum_{j=0}^{q-1} \left((1 + (m + 4j + 1)d)^2 - (1 + (m + 4j + 2)d)^2 \right. \\ \left. - (1 + (m + 4j + 3)d)^2 + (1 + (m + 4j + 4)d)^2 \right).$$

Similarly, for $q < 0$ we have

$$4d^2q = \sum_{j=0}^{-q-1} \left(- (1 + (m + 4j + 1)d)^2 + (1 + (m + 4j + 2)d)^2 \right. \\ \left. + (1 + (m + 4j + 3)d)^2 - (1 + (m + 4j + 4)d)^2 \right).$$

Consequently, S can be written in the desired form.

