OC111. Soit x, y et z trois nombres réels positifs. Démontrez que

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4\,.$$

OC112. Déterminer toutes les paires de nombres naturels (a, b) tels que

$$\text{pgcd}(a, b) + 9\,\text{ppcm}(a, b) + 9(a + b) = 7ab\,.$$

OC113. Démontrer que parmi n'importe quels n sommets d'un polygone régulier à $(2n - 1)$ sommets, où $n \geq 3$, on peut en tirer 3 qui forment un triangle isocèle.

OC114. Soit ABC un triangle scalène. Son cercle inscrit touche BC, AC et AB aux points D, E et F respectivement. Soient L, M et N les points symétriques à D, E et F par rapport à EF, FD et DE respectivement. La ligne AL intersecte BC en P, la ligne BM intersecte CA en Q et la ligne CN intersecte AB en R. Démontrer que P, Q et R sont colinéaires.

OC115. Déterminer le plus petit entier positif n pour lequel il existe un entier positif k tel que les 2012 dernières positions décimales de n^k sont toutes 1.

OLYMPIAD SOLUTIONS

OC51. Determine all pairs (a, b) of nonnegative integers so that $a^b + b$ divides $a^{2b} + 2b$. Note, for this problem $0^0 = 1$.
(*Originally question 4 from the second day of Austrian Mathematical Olympiad.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Oliver Geupel, Brühl, NRW, Germany ; Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA ; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Cománeşti, Romania. We give the solution of Wang.

The only solutions are $(a, 0), (0, b)$ for $a, b \geq 0$ and $(a, b) = (2, 1)$. Since $(a, 0)$ and $(0, b)$ clearly satisfy the equation, it remains to show that if $a \geq 1$ and $b \geq 1$ then $(a, b) = (2, 1)$.

Suppose that (a, b) is a solution where $a, b \geq 1$. If $a = 1$ then the condition becomes $1 + b \mid 1 + 2b$ which is impossible since $1 + b < 1 + 2b < 2(1 + b)$. Hence we have $a \geq 2$.

As $a^b + b \mid a^{2b} + 2b$ we have

$$\begin{aligned} a^b + b &= \gcd(a^b + b, a^{2b} + 2b) = \gcd(a^b + b, a^b(a^b + b) - (a^{2b} + 2b)) \\ &= \gcd(a^b + b, ba^b - 2b) = \gcd(a^b + b, b(a^b + b) - (ba^b - 2b)) \\ &= \gcd(a^b + b, b^2 + 2b) \end{aligned}$$

Thus

$$a^b + b \le b^2 + 2b\,.$$

Hence

$$a^b \le b^2 + b\,. \tag{1}$$

We claim that (1) cannot hold if $a \ge 3$ by proving by induction that

$$3^b > b^2 + b$$

for all $b \ge 1$. This is clear when $b = 1$.

Suppose that $3^b > b^2 + b$ for some $b \ge 1$. Then

$$\begin{aligned} 3^{b+1} > 3(b^2 + b) &= b^2 + 2b + b^2 + b + b^2 \\ &\ge b^2 + 2b + 1 + b + 1 = (b+1)^2 + (b+1), \end{aligned}$$

completing the induction.

It remains to consider the case when $a = 2$. Using the same induction argument, it can be proved easily that

$$2^b > b^2 + b$$

for all $b \ge 5$, which contradicts (1). Thus we only need to check the cases $(a, b) \in \{(2, 1), (2, 2), (2, 3), (2, 4)\}$, and a direct computation shows that only $(2, 1)$ yields a solution.

This completes the proof.

OC52. Let d, d' be two divisors of n with $d' > d$. Prove that

$$d' > d + \frac{d^2}{n}\,.$$

(*Originally question 1 from the Russia National Olympiad 2011: Grade 11.*)

Solved by Michel Bataille, Rouen, France ; Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Marian Dincă, Bucharest, Romania ; Oliver Geupel, Brühl, NRW, Germany ; Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA ; Kim Uyen Truong, California State University, Fullerton, CA, USA ; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Truong.

Since d and d' are divisors of n there exist some a, b so that

$$ad' = bd = n\,.$$

As $d' > d$ we have $a < b$ and hence $a \le b - 1$. Thus

$$\begin{aligned} d' > d + \frac{d^2}{n} &\Leftrightarrow \frac{n}{a} > \frac{n}{b} + \frac{n}{b^2} \\ &\Leftrightarrow \frac{1}{a} > \frac{1}{b} + \frac{1}{b^2} \\ &\Leftrightarrow b^2 > ab + a = a(b+1)\,. \end{aligned}$$

This last inequality holds as

$$a(b+1) \le (b-1)(b+1) = b^2 - 1 < b^2\,.$$

Thus,

$$d' > d + \frac{d^2}{n}\,.$$

OC53. Find all the polynomials $P(x) \in \mathbb{R}[x]$ so that $P(a) \in \mathbb{Z}$ implies $a \in \mathbb{Z}$.
(*Originally question 4 from the 2011 Singapore National Olympiad.*)

Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany and Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA. We give the solution of Curtis.

Case 1: $\deg(P) = 0$. Then, there exists a constant c so that $P(x) = c$ for all real x. Then this polynomial has the desired property if and only if c is not an integer. [If c is not an integer, then P has the desired property by vacuity].

Case 2: $\deg(P) = 1$. Then $P(x) = ax + b$ for some real numbers a ($\neq 0$) and b. As the range of P is the real numbers, for each integer m there is an integer x_m so that $P(x_m) = m$. Then

$$ax_m + b = m \quad \text{and} ax_{m-1} + b = m - 1,$$

which implies that

$$x_{m-1} = \frac{(m-1) - b}{a} \quad \text{and} x_m = \frac{m-b}{a}.$$

Subtracting, we get that

$$\frac{1}{a} = x_m - x_{m-1} \in \mathbb{Z}\,.$$

Thus $a = \frac{1}{k}$ with k an integer. Then

$$x_m = \frac{m-b}{a} = km - kb \Rightarrow kb = x_m - km \in \mathbb{Z}\,.$$

Let $l = kb \in \mathbb{Z}$, then we get

$$P(x) = \frac{x+l}{k}\,.$$

It is easy to see that this polynomial has the desired property.

Case 3: $\deg(P) \geq 2$. We claim there is no such polynomial in this case.

Since for any polynomial P which satisfies the condition, $-P$ also satisfies the condition, without loss of generality we can assume that the leading coefficient of P is positive. We then have $\lim\limits_{x\to\infty} P'(x) = \infty$, and so there exists some x_0 so that $P'(x) \geq 2$ for all $x \geq x_0$.

Moreover, as $\lim\limits_{x\to\infty} P(x) = \infty$, by the Intermediate Value Theorem there exists some M so that the interval

$$[M, \infty) \subset P((x_0, \infty))\,.$$

Thus, for every $m > M$ there exist two distinct integers $u, v > x_0$ so that

$$P(u) = m\,;\; P(v) = m + 1\,.$$

By the Mean Value Theorem, there exists some c_m between u, v so that

$$P'(c_m) = \frac{P(u) - P(v)}{u - v} = \frac{1}{u - v} < 1\,.$$

But as $c_m > x_0$ we also have $P'(c_m) \geq 2$, a contradiction. Thus there is no polynomial of degree 2 or higher satisfying this condition.

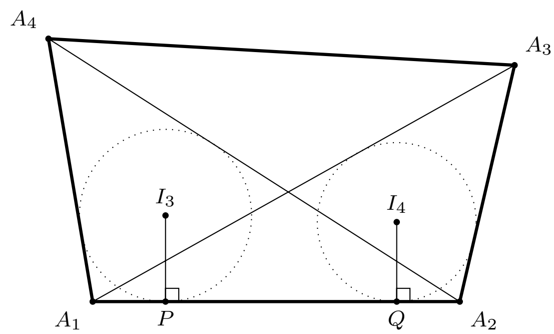
In summary, a polynomial P satisfies this condition if and only if either P is a nonintegral constant or $P(x) = \frac{x+l}{k}$ for some integers l, k with $k \neq 0$.

OC54. Given four points in the plane so that the incircles of the four triangles formed by three of the four points are equal, prove that the four triangles are equal. (*Originally question 5 from the 2011 Japan National Olympiad.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We show that the four points are the vertices of a rectangle.

Let the four points be A_1, A_2, A_3, and A_4. Let a_{ik} denote the length of the segment A_iA_k. Let I_1, I_2, I_3, and I_4 denote the incentres of the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. If the convex hull $\mathcal{C}$ of the four points is a triangle, say, triangle $A_1A_2A_3$, then A_4 is an interior point of that triangle, which implies that the inradius of triangle $A_1A_2A_3$ is greater than the inradius of triangle $A_1A_2A_4$, a contradiction. Hence, $\mathcal{C}$ is a quadrilateral, say, the convex quadrilateral $A_1A_2A_3A_4$. Let P and Q be the orthogonal projections of I_3 and I_4 on the line A_1A_2. Because triangles $A_1A_2A_4$ and $A_1A_2A_3$ have congruent incircles, the quadrilateral PQI_4I_3 is a rectangle.



We obtain

$$\begin{aligned} I_3I_4 = PQ &= A_1A_2 - A_1P - A_2Q \\ &= a_{12} - \frac{a_{12}+a_{14}-a_{24}}{2} - \frac{a_{12}+a_{23}-a_{13}}{2} \\ &= \frac{a_{13}+a_{24}-a_{14}-a_{23}}{2}. \end{aligned}$$

Similarly,

$$I_1I_2 = \frac{a_{13}+a_{24}-a_{14}-a_{23}}{2},$$

thus $I_1I_2 = I_3I_4$. Analogously, $I_4I_1 = I_2I_3$. Hence, the quadrilateral $I_1I_2I_3I_4$ is a parallelogram.

We obtain $A_1A_2 \parallel I_3I_4 \parallel I_1I_2 \parallel A_3A_4$ and similarly $A_1A_4 \parallel A_2A_3$. Thus, the quadrilateral $A_1A_2A_3A_4$ is a parallelogram. Therefore,

$$\begin{aligned} a_{12}+a_{14}+a_{24} &= \frac{2[A_1A_2A_4]}{I_3P} = \frac{2[A_1A_2A_3]}{I_4Q} \\ &= a_{12}+a_{23}+a_{13} = a_{12}+a_{14}+a_{13}. \end{aligned}$$

Consequently, $A_2A_4 = a_{24} = a_{13} = A_1A_3$.

This shows that the quadrilateral $A_1A_2A_3A_4$ is a rectangle.

OC55. Let d be a positive integer. Show that for every integer S there exists an integer $n > 0$ and a sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, where for any $k, \epsilon_k = 1$ or $\epsilon_k = -1$, such that

$$S = \epsilon_1(1+d)^2 + \epsilon_2(1+2d)^2 + \epsilon_3(1+3d)^2 + \cdots + \epsilon_n(1+nd)^2.$$

(*Originally question 5 from 2011 Canadian Mathematical Olympiad.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

Let S be any integer. We are going to show that S can be written in the required form.

Let m be an integer such that $m \geq 4d^3$ and that the number

$$S_0 = -\sum_{k=1}^{m} (1+kd)^2\,,$$

has the same parity as S. Define now

$$S_k = S_{k-1} + 2(1+2kd^2)^2$$

for $k = 1, 2, 3, \ldots, 2d^2 - 1$. We have

$$S_k \equiv S_{k-1} + 2 \pmod{4d^2}\,.$$

Hence, the sequence

$$S_0, S_1, S_2, \ldots, S_{2d^2-1}$$

includes a number that has the same remainder modulo $4d^2$ as S. Let S' be that number.

Then, S' can be written in the required form and there is an integer q such that

$$S = S' + 4d^2 q\,.$$

Observe that for each integer k we have

$$(1+kd)^2 - (1+(k+1)d)^2 - (1+(k+2)d)^2 + (1+(k+3)d)^2 = 4d^2\,.$$

Therefore, for each $q \geq 0$ we have

$$\begin{aligned} 4d^2 q = \sum_{j=0}^{q-1} \Big(&(1+(m+4j+1)d)^2 - (1+(m+4j+2)d)^2 \\ &- (1+(m+4j+3)d)^2 + (1+(m+4j+4)d)^2 \Big). \end{aligned}$$

Similarly, for $q < 0$ we have

$$\begin{aligned} 4d^2 q = \sum_{j=0}^{-q-1} \Big(&-(1+(m+4j+1)d)^2 + (1+(m+4j+2)d)^2 \\ &+ (1+(m+4j+3)d)^2 - (1+(m+4j+4)d)^2 \Big). \end{aligned}$$

Consequently, S can be written in the desired form.