THE OLYMPIAD CORNER

No. 309

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file* <u>*and*</u> *a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* `crux-olympiad@cms.math.ca`*. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on* <u>*each*</u> *solution, and each solution should start on a separate page.*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 May 2014***, although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet, of l'Université Saint-Boniface in Winnipeg, for translations of the problems.

OC111. Let x, y and z be positive real numbers. Show that

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4\,.$$

OC112. Find all pairs of natural numbers (a, b) such that

$$\gcd(a, b) + 9\,\mathrm{lcm}(a, b) + 9(a + b) = 7ab\,.$$

OC113. Prove that among any n vertices of a regular $(2n-1)$–gon, where $n \geq 3$, we can find 3 which form an isosceles triangle.

OC114. Let ABC be a scalene triangle. Its incircle touches BC, AC, AB at D, E, F respectively. Let L, M, N be the symmetric points of D, E, F with respect to EF, FD, and DE, respectively. The line AL intersects BC at P, the line BM intersects CA at Q, and the line CN intersects AB at R. Prove that P, Q, R are collinear.

OC115. Find the smallest positive integer n for which there exists a positive integer k such that the last 2012 decimal digits of n^k are all 1's.

OC111. Soit x, y et z trois nombres réels positifs. Démontrez que

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4\,.$$

OC112. Déterminer toutes les paires de nombres naturels (a, b) tels que

$$\operatorname{pgcd}(a, b) + 9\operatorname{ppcm}(a, b) + 9(a + b) = 7ab\,.$$

OC113. Démontrer que parmi n'importe quels n sommets d'un polygone régulier à $(2n - 1)$ sommets, où $n \geq 3$, on peut en tirer 3 qui forment un triangle isocèle.

OC114. Soit ABC un triangle scalène. Son cercle inscrit touche BC, AC et AB aux points D, E et F respectivement. Soient L, M et N les points symétriques à D, E et F par rapport à EF, FD et DE respectivement. La ligne AL intersecte BC en P, la ligne BM intersecte CA en Q et la ligne CN intersecte AB en R. Démontrer que P, Q et R sont colinéaires.

OC115. Déterminer le plus petit entier positif n pour lequel il existe un entier positif k tel que les 2012 dernières positions décimales de n^k sont toutes 1.

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OC51. Determine all pairs (a, b) of nonnegative integers so that $a^b + b$ divides $a^{2b} + 2b$. Note, for this problem $0^0 = 1$.
(*Originally question 4 from the second day of Austrian Mathematical Olympiad.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Oliver Geupel, Brühl, NRW, Germany ; Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA ; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Cománeşti, Romania. We give the solution of Wang.

The only solutions are $(a, 0), (0, b)$ for $a, b \geq 0$ and $(a, b) = (2, 1)$. Since $(a, 0)$ and $(0, b)$ clearly satisfy the equation, it remains to show that if $a \geq 1$ and $b \geq 1$ then $(a, b) = (2, 1)$.

Suppose that (a, b) is a solution where $a, b \geq 1$. If $a = 1$ then the condition becomes $1 + b \mid 1 + 2b$ which is impossible since $1 + b < 1 + 2b < 2(1 + b)$. Hence we have $a \geq 2$.