FOCUS ON ...

No. 5

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Inequalities *via* Lagrange Multipliers

Introduction

Problems requiring a proof of an inequality are frequent and often difficult. For constrained inequalities, the method of Lagrange multipliers is generally put aside at first, its application being considered as delicate. Without advocating any systematic use, we propose a few examples where, with some care, the method leads to a simple solution. We will restrict ourselves to three real variables and one equality constraint, in which case the theorem to be used is as follows: let f, g be continuously differentiable functions on an open subset U of $\mathbb{R}^3$ and $(a,b,c) \in F = \{(x,y,z) \in U : g(x,y,z) = 0\}$. If $f(a,b,c) \leq f(x,y,z)$ [or $f(a,b,c) \geq f(x,y,z)$] for all $(x,y,z) \in F$ and $(\partial_1 g(a,b,c), \partial_2 g(a,b,c), \partial_3 g(a,b,c)) \neq (0,0,0)$, then for some real number λ, we have $\partial_i f(a,b,c) = \lambda \partial_i g(a,b,c)$ for $i = 1,2,3$ (for a proof, see [1] for example). Here ∂_i denotes the partial derivative with respect to the i^{th} variable.

A maximum on the boundary

Our first example, Bilkent University Problem of the Month in November 2007, asked for the maximal value of $x^4y + y^4z + z^4x$ when x, y, z are nonnegative real numbers satisfying $x + y + z = 5$. Note that the continuous function

$$f : (x,y,z) \mapsto f(x,y,z) = x^4y + y^4z + z^4x$$

attains its maximum M on the compact set

$$K = \{(x,y,z) \in \mathbb{R}^3 \mid x,y,z \geq 0 \quad \text{and} \quad x + y + z = 5\}$$

and we are to find the value of M. First, we examine what occurs when (x,y,z) is on the boundary of K, say $z = 0,\ x + y = 5$. Then,

$$\begin{aligned} f(x,y,z) &= f(x,5-x,0) = x^4(5-x) \\ &= 4^4 \cdot \frac{x}{4} \cdot \frac{x}{4} \cdot \frac{x}{4} \cdot \frac{x}{4}(5-x) \leq 4^4 \cdot \left(\frac{1}{5}(4 \times \frac{x}{4} + 5 - x)\right)^5 = 4^4 \end{aligned}$$

with equality when $x = 4$. Thus, the maximum value of f on the boundary of K is 256 and $M \geq 256$.

Now, assume that M is attained at an interior point (a,b,c) of K. Since the interior of K is open and the constraint is $x + y + z - 5 = 0$, there would exist a Lagrange multiplier λ such that

$$\partial_1 f(a,b,c) = \partial_2 f(a,b,c) = \partial_3 f(a,b,c) = \lambda$$

that is,

$$4a^3b + c^4 = \lambda,\ 4b^3c + a^4 = \lambda,\ 4c^3a + b^4 = \lambda.$$

Therefore, on the one hand, we would have $\lambda(a+b+c) = 5f(a,b,c)$ that is, $\lambda = M$ and on the other hand,

$$3\lambda = a^4 + b^4 + c^4 + 4a^3b + 4b^3c + 4c^3a < (a+b+c)^4 = 5^4.$$

As a result, $M < \frac{5^4}{3}$, in contradiction with $M \geq 256$. Thus, M is attained on the boundary of K and so $M = 4^4 = 256$. [For an alternative solution, see [2]].

A maximum in the interior

A slightly different example is provided by problem **3032** [2005 : 174, 177; 2006 : 190]:

Prove that

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca} \leq \frac{9}{2}$$

whenever a, b, c *are nonnegative real numbers such that* $a^2+b^2+c^2 = 1$.

The featured solution by the proposer Vasile Cîrtoaje rests upon clever manipulations and identities. The method of Lagrange multipliers allows a different approach.

Let

$$f(x,y,z) = \frac{1}{1-xy} + \frac{1}{1-yz} + \frac{1}{1-zx}$$

and

$$K = \{(x,y,z) \in \mathbb{R}^3 : x,y,z \geq 0\ ,\ x^2+y^2+z^2 = 1\}.$$

The continuous function f reaches its maximum M on the compact set K and $M \geq f(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = \frac{9}{2}$. Moreover, if $(0,y,z) \in K$, then $f(0,y,z) = 2 + \frac{1}{1-yz} \leq 4 < M$. It follows that M is attained at (a,b,c), an interior point of K. We show that $a = b = c$, from which the result follows at once.

Since $f(x,y,z)$ is invariant under any permutation of x, y, z, we may suppose that $a \leq b \leq c$ so that $ab \leq ac \leq bc$. Since $bc < 1/3$ implies $f(a,b,c) < 3 \times \frac{3}{2} \leq M$, we must have $bc \geq 1/3$ and so $ab \leq 1/3, ab + ac \leq 2/3$ (note that $ab + ac + bc \leq a^2 + b^2 + c^2 = 1$). Now, there exists a Lagrange multiplier λ such that

$$\frac{b}{(1-ab)^2} + \frac{c}{(1-ac)^2} = 2\lambda a; \qquad \frac{a}{(1-ab)^2} + \frac{c}{(1-bc)^2} = 2\lambda b;$$

$$\frac{b}{(1-bc)^2} + \frac{a}{(1-ac)^2} = 2\lambda c;$$

from which we readily deduce

$$\frac{c^2-b^2}{bc(1-bc)^2} = a^2(\phi(ac) - \phi(ab)) \tag{1}$$

where the function ϕ is defined by $\phi(t) = \frac{1}{t(1-t)^2}$. It follows that

$$\phi(ab) \leq \phi(ac). \tag{2}$$

It is easily checked that ϕ is decreasing on $(0, 1/3]$ and increasing on $[1/3, 1)$ and that $\phi(2/3 - t) < \phi(t)$ for $t \in (0, 1/3)$. These properties and (2) forbid $ac > 1/3$, hence $ab \leq ac \leq 1/3$ and so $\phi(ab) \geq \phi(ac)$ and $b = c$ (by (1)). Now, from

$$\lambda = \frac{b}{a(1-ab)^2} = \frac{a}{2b(1-ab)^2} + \frac{1}{2(1-b^2)^2}$$

and using $a^2 = 1 - b^2 - c^2 = 1 - 2b^2$, we deduce

$$(4b^2-1)(1-b^2)^2 = \frac{1}{\phi(ab)} \leq \frac{1}{\phi(1/3)} = \frac{4}{27}.$$

But a quick study of the function $t \mapsto \psi(t) = (4t-1)(1-t)^2$ shows that $\psi(t) \geq \psi(1/3) = \frac{4}{27}$ for $t \in [\frac{1}{3}, \frac{1}{2})$ with equality only for $t = 1/3$. It follows that $b^2 = 1/3$ and so $a = b = c = 1/\sqrt{3}$.

Two exercises

(a) In reference to problem **2843** [2003 : 463; 2004 : 250], for $x, y, z > 0$, let

$$f(x, y, z) = (1-x)(1-y)(1-z)$$

and

$$g(x, y, z) = 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 4 - \left(\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}\right).$$

Show that $(a, b, c) = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ satisfies the constraint $g(x, y, z) = 0$ and $\partial_i f(a, b, c) = \lambda \partial_i g(a, b, c)$ $(i = 1, 2, 3)$ for some λ but $f(a, b, c)$ is not an extremum of f under the constraint.

(b) A part of problem **2787** [2002 : 460; 2003 : 477] was the inequality

$$\frac{1}{1-\left(\frac{x+y}{2}\right)^2} + \frac{1}{1-\left(\frac{y+z}{2}\right)^2} + \frac{1}{1-\left(\frac{z+x}{2}\right)^2} \leq \frac{11}{3}$$

when $x + y + z = 1$ and $x, y, z \geq 0$. Prove this inequality with the method of Lagrange multipliers. (Hint : under the constraint, the left-hand side of the inequality is $h(x) + h(y) + h(z)$ where $h(t) = \frac{1}{3-t} + \frac{1}{1+t}$ and the derivative h' is strictly monotone.)

References

[1] J.E. Marsden, A.J. Tromba, *Vector Calculus*, Freeman, 1981, p. 217.

[2] Bilkent University Problem of the Month, November 2007(solution), `www.fen.bilkent.edu.tr/~cvmath/Problem/0711a.pdf`