THE CONTEST CORNER

No. 11

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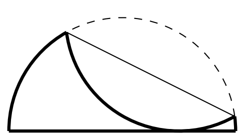
The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_CCProblemNumber *(example Doe_Jane_CC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* `crux-contest@cms.math.ca`*. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page.*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 May 2014***, although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the Solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC51. A semicircular piece of paper with radius 2 is creased and folded along a chord so that the arc is tangent to the diameter as shown in the diagram. If the contact point of the arc divides the diameter in the ratio 3 : 1, determine the length of the crease.



CC52. There are some marbles in a bowl. Alphonse, Beryl and Colleen each take turns removing one or two marbles from the bowl, with Alphonse going first, then Beryl, then Colleen, then Alphonse again, and so on. The player who takes the last marble from the bowl is the loser, and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can Beryl and Colleen work together and force Alphonse to lose?

CC53. Determine an infinite family of quadruples (a, b, c, d) of positive integers, each of which is a solution to $a^4 + b^5 + c^6 = d^7$.

CC54. Let k, l, m, n be positive integers such that $k+l+m \geq n$. Prove the following relation for binomial coefficients

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}.$$

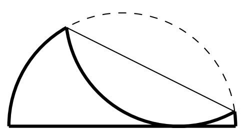
The summation in the left-hand side runs over all ordered partitions of n into three integers p, q, r such that $0 \leq p \leq k$, $0 \leq q \leq l$, $0 \leq r \leq m$.

CC55. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

. . . .

CC51. Un morceau de papier de forme semi-circulaire, de rayon 2, est plié le long d'une corde de manière que l'arc soit tangent au diamètre, comme dans la figure. Sachant que le point de contact de l'arc et du diamètre divise le diamètre dans un rapport de 3 : 1, déterminer la longueur du pli.



CC52. Un bol contient un nombre de billes. Alain, Bianca et Carla enlèvent tour à tour une ou deux billes du bol, en commençant par Alain, suivi de Bianca, puis de Carla, puis d'Alain et ainsi de suite. Le joueur qui enlève la dernière bille perd la joute, tandis que les deux autres sont gagnants. S'il y a N billes dans le bol au départ, pour quelles valeurs de N Bianca et Carla peuvent-elles travailler ensemble pour qu'Alain soit toujours perdant ?

CC53. Déterminer une famille infinie de quadruplets (a, b, c, d) d'entiers strictement positifs, chaque quadruplet étant une solution de l'équation $a^4 + b^5 + c^6 = d^7$.

CC54. Soit k, l, m et n des entiers strictement positifs tels que $k+l+m \geq n$. Démontrer la relation suivante qui comporte des coefficients binômiaux :

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}$$

La sommation du membre de gauche se fait sur toutes les partitions ordonné de n en trois entiers p, q, r de manière que $0 \leq p \leq k$, $0 \leq q \leq l$ et $0 \leq r \leq m$.

CC55. Soit α, β, γ les racines de l'équation $x^3 - x - 1 = 0$. Évaluer l'expression

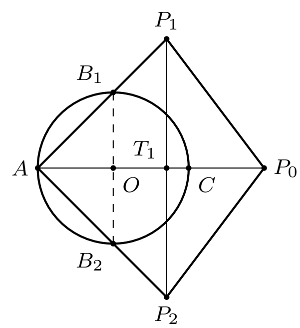
$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

CONTEST SOLUTIONS

CC1. A circle has centre O, diameter AC, and radius 1. A chord is drawn from A to an arbitrary point B (different from A) on the circle and extended to the point P with $BP = 1$. Thus P can take many positions. Let S be the set of points P. Determine whether or not there is a circle on which all points of S lie.
(*Originally question B3 part c) from the 2010 Sun Life Financial Canadian Open Mathematics Challenge.*)

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comăneşti, Romania. One incorrect solution was received. We give the solution of Zvonaru.

Let P_0 be the symmetric of O with respect to C, then P_0 is the position of P when B is located at C. Let B_1B_2 be a diameter of the given circle such that $B_1B_2 \perp AC$ and let P_1 and P_2 be the locations of P when B is located at B_1 and B_2 respectively. Let T_1 be the point of intersection of AP_0 and P_1P_2.



We obtain:

$$AB_1 = \sqrt{2}, \quad AP_1 = 1 + \sqrt{2},$$
$$AT_1 = P_1T_1 = \frac{1+\sqrt{2}}{\sqrt{2}} = \frac{2+\sqrt{2}}{2}.$$

Hence

$$T_1P_0 = AP_0 - AT_1 = 3 - \frac{2+\sqrt{2}}{2} = \frac{4-\sqrt{2}}{2}$$

and by the Pythagorean theorem we deduce that

$$P_0P_1^2 = P_1T_1^2 + T_1P_0^2 = \left(\frac{2+\sqrt{2}}{2}\right)^2 + \left(\frac{4-\sqrt{2}}{2}\right)^2$$
$$= \frac{6+4\sqrt{2}+18-8\sqrt{2}}{4} = 6-\sqrt{2}.$$

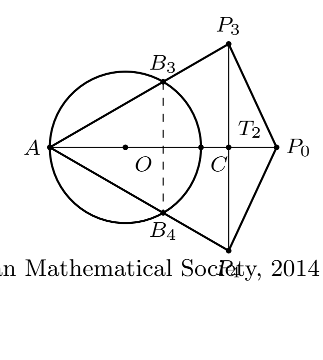
If R_1 is the circumradius of $\Delta P_0P_1P_2$, then

$$R_1 = \frac{P_1P_2 \cdot P_0P_1 \cdot P_2P_0}{4 \cdot \text{Area}(\Delta P_0P_1P_2)} = \frac{P_1P_2 \cdot P_1P_0^2}{4\left(P_1P_2 \cdot \frac{T_1P_0}{2}\right)} = \frac{6-\sqrt{2}}{4-\sqrt{2}} = \frac{11+\sqrt{2}}{7}.$$

Similarly, we let B_3 and B_4 be on the circle, so that

$$\angle B_3AC = \angle B_4AC = 30^\circ;$$

let P_3 and P_4 be the corresponding locations of P, and let T_2 be the point of intersection of P_3P_4 and AP_0.



The triangle AP_3P_4 is equilateral and we have:

$$AB_3 = \sqrt{3}, \quad AP_3 = 1+\sqrt{3},$$
$$AT_2 = AP_3 \cdot \frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2},$$
$$T_2P_0 = 3 - \frac{3+\sqrt{3}}{2} = \frac{3-\sqrt{3}}{2}\,.$$

From the Law of Cosines applied to ΔAP_0P_3, we obtain

$$\begin{aligned} P_0P_3^2 &= AP_0^2 + AP_3^2 - 2\cdot AP_0 \cdot AP_3 \cdot \cos 30^\circ \\ &= 9+4+2\sqrt{3} - 2\cdot\frac{\sqrt{3}}{2}\cdot 3\cdot(1+\sqrt{3}) \\ &= 4-\sqrt{3}\,. \end{aligned}$$

If R_2 is the circumradius of $\Delta P_0P_1P_2$, then it results that

$$R_2 = \frac{P_3P_4 \cdot P_3P_0^2}{4\left(P_3P_4\cdot\frac{T_2P_0}{2}\right)} = \frac{4-\sqrt{3}}{3-\sqrt{3}} = \frac{9+\sqrt{3}}{6}\,.$$

Since $R_1 \neq R_2$, it follows that there is not a circle on which all points of S lie.

CC2. Let $f:\mathbb{R}^2\to\mathbb{R}$. Suppose that f is continuous and that $\int_0^1 f(a+tu)dt = 0$ for every point $a\in\mathbb{R}^2$ and every vector $u\in\mathbb{R}^2$ with $\|u\| = 1$. Show that f is constant.
(*Originally question # 4 from the 2012 University of Waterloo Big E Contest.*)

One incorrect solution was received. The problem remains open

CC3. All three sides of a right triangle are integers. Prove that the area of the triangle: is also an integer; is divisible by 3; and is even.
(*Originally question # 10 from the 2010 Manitoba Mathematical Competition.*)

Solved by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Daniel Văcaru, Piteşti, Romania; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Haley Williams, student, Auburn University at Montgomery, Montgomery, AL, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Apostolopoulos modified by the editor.

Suppose the legs of the triangle have lengths a and b, and the hypotenuse has length c. The area of the triangle is $\frac{ab}{2}$, so in order to prove the area is even and divisible by 3, it suffices to show $4|ab$ and $3|ab$.

In this light, let $d\in\{3,4\}$. Observe that for any integer x,

$$x^2 \equiv 0,1 \pmod{d} \tag{1}$$

Thus, if both a and b are not divisible by d, then $a^2 \equiv 1 \pmod{d}$ and $b^2 \equiv 1 \pmod{d}$, and hence

$$c^2 = a^2 + b^2 \equiv 2 \pmod{d},$$

which contradicts (1). Therefore, for $d \in \{3, 4\}$, $d|a$ or $d|b$, implying $d|ab$.

CC4. Suppose that $n \geq 3$. A sequence $a_1, a_2, a_3, \ldots, a_n$ of n integers, the first m of which are equal to -1 and the remaining $p = n - m$ of which are equal to 1, is called an MP sequence. Consider all of the products $a_i a_j a_k$ (with $i < j < k$) that can be calculated using the terms from an MP sequence $a_1, a_2, a_3, \ldots, a_n$. Determine the number of pairs (m, p) with $1 \leq m \leq p \leq 1000$ and $m + p \geq 3$ for which exactly half of these products are equal to 1.
(*Originally question B3 part b) from the 2011 Canadian Senior Mathematics Contest.*)

Solved together by Billy Jin, Waterloo Collegiate Institute, Waterloo, ON and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. One incorrect solution was received.

Call an MP sequence *balanced* if it satisfies the specified condition. For a given MP sequence let M and P denote the number of triples (a_i, a_j, a_k) where $i < j < k$ such that $a_i a_j a_k = -1$ or 1, respectively. Then the condition $M = P$ is successively equivalent to:

$$\binom{m}{3} + \binom{m}{1}\binom{p}{2} = \binom{p}{3} + \binom{p}{1}\binom{m}{2}$$

$$\frac{m(m-1)(m-2)}{6} + \frac{mp(p-1)}{2} = \frac{p(p-1)(p-2)}{6} + \frac{pm(m-1)}{2}$$

$$3mp(m-p) = m(m-1)(m-2) - p(p-1)(p-2)$$

$$3mp(m-p) = m^3 - p^3 - 3(m^2 - p^2) + 2(m-p) \qquad (1)$$

Hence by setting $m = p = c$ where $c \in \mathbb{N}$ and $1 < c \leq 1000$, we get 999 balanced MP sequences.

Suppose now that $m \neq p$. Then dividing both sides of (1) by $m - p$ yields

$$3mp = m^2 + mp + p^2 - 3(m+p) + 2$$

$$p^2 - (2m+3)p + (m^2 - 3m + 2) = 0 \qquad (2)$$

Since p is an integer, the discriminant, D, of (2) must be a perfect square.

Now, $D = (2m+3)^2 - 4(m^2 - 3m + 2) = 24m + 1$, so

$$24m + 1 = k^2 \text{ for some } k \in \mathbb{N} \text{ with } k \geq 5 \qquad (3)$$

Note that $k^2 \equiv 1 \pmod{8}$ (if and only if k is odd) and $k^2 \equiv 1 \pmod{3}$ (if and only if $k \equiv 1$ or $2 \pmod{3}$). Hence any odd $k \geq 5$ such that $k \not\equiv 0 \pmod{3}$ would satisfy (3) for some m which would yield an admissible pair (m, p) provided the condition $1 \leq m \leq p \leq 1000$ holds. To satisfy this condition we solve (2) and choose $p = \frac{2m+3+\sqrt{D}}{2} = m + \frac{k+3}{2} > m$.

Since $p \le 1000$ we have $\frac{k^2-1}{24} + \frac{k+3}{2} \le 1000$ which implies $k^2 + 12k + 35 \le 24000$ or $(k+6)^2 \le 24001$, so $k + 6 \le 154$ or $k \le 148$. Hence it remains to count the number, q, of odd integers k with $5 \le k \le 148$ which are not divisible by 3. Among the 72 odd integers in the set $\{5, 6, \cdots, 148\}$, exactly 24 of them are divisible by 3 since $k = 3 + 6d \le 147$ implies $d \le 24$. Hence, $q = 72 - 24 = 48$.

Therefore, the required number is $999 + 48 = 1047$.

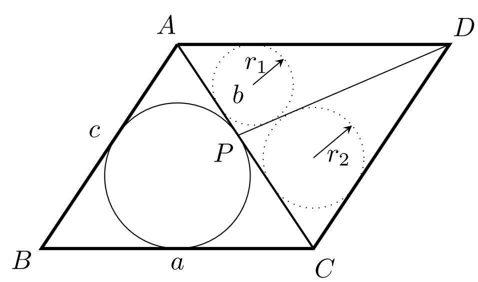
CC5. Let $ABCD$ be a parallelogram. We draw in the diagonal AC. A circle is drawn inside ΔABC tangent to all three sides and touches side AC at a point P. Draw in the line DP. A circle of radius r_1 is drawn inside ΔDAP tangent to all three sides and a circle of radius r_2 is drawn inside ΔDCP tangent to all three sides. Prove that

$$\frac{r_1}{r_2} = \frac{AP}{PC}.$$

(*Originally question C3 part b) from the 2012 Sun Life Financial Canadian Open Mathematics Challenge.*)

Solved by Daniel Văcaru, Piteşti, Romania; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

We denote $a = BC$, $b = AC$, $c = AB$.



It is well known that

$$AP = \frac{b+c-a}{2}, \quad \text{and} \quad PC = \frac{a+b-c}{2}.$$

Since

$$r_1 = \frac{\text{area}(\Delta APD)}{\frac{AP+PD+DA}{2}}, \quad r_2 = \frac{\text{area}(\Delta PCD)}{\frac{PC+CD+DP}{2}}, \quad \text{and} \quad \frac{\text{area}(\Delta APD)}{\text{area}(\Delta PCD)} = \frac{AP}{PC}.$$

Thus

$$\frac{r_1}{r_2} = \frac{AP(PC+CD+PD)}{PC(AP+PD+AD)} = \frac{AP}{PC} \times \frac{\frac{a+b-c}{2} + c + PD}{\frac{b+c-a}{2} + a + PD} = \frac{AP}{PC} \times \frac{\frac{a+b+c}{2} + PD}{\frac{a+b+c}{2} + PD},$$

and therefore $\frac{r_1}{r_2} = \frac{AP}{PC}$.