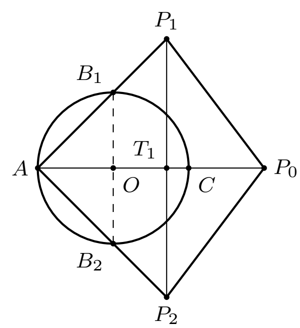
CONTEST SOLUTIONS

CC1. A circle has centre O, diameter AC, and radius 1. A chord is drawn from A to an arbitrary point B (different from A) on the circle and extended to the point P with $BP = 1$. Thus P can take many positions. Let S be the set of points P. Determine whether or not there is a circle on which all points of S lie.
(*Originally question B3 part c) from the 2010 Sun Life Financial Canadian Open Mathematics Challenge.*)

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comăneşti, Romania. One incorrect solution was received. We give the solution of Zvonaru.



Let P_0 be the symmetric of O with respect to C, then P_0 is the position of P when B is located at C. Let B_1B_2 be a diameter of the given circle such that $B_1B_2 \perp AC$ and let P_1 and P_2 be the locations of P when B is located at B_1 and B_2 respectively. Let T_1 be the point of intersection of AP_0 and P_1P_2.

We obtain:

$$AB_1 = \sqrt{2}, \quad AP_1 = 1 + \sqrt{2},$$
$$AT_1 = P_1T_1 = \frac{1+\sqrt{2}}{\sqrt{2}} = \frac{2+\sqrt{2}}{2}.$$

Hence

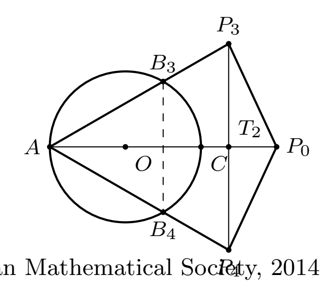
$$T_1P_0 = AP_0 - AT_1 = 3 - \frac{2+\sqrt{2}}{2} = \frac{4-\sqrt{2}}{2}$$

and by the Pythagorean theorem we deduce that

$$P_0P_1^2 = P_1T_1^2 + T_1P_0^2 = \left(\frac{2+\sqrt{2}}{2}\right)^2 + \left(\frac{4-\sqrt{2}}{2}\right)^2$$
$$= \frac{6+4\sqrt{2}+18-8\sqrt{2}}{4} = 6-\sqrt{2}.$$

If R_1 is the circumradius of $\Delta P_0P_1P_2$, then

$$R_1 = \frac{P_1P_2 \cdot P_0P_1 \cdot P_2P_0}{4 \cdot \text{Area}(\Delta P_0P_1P_2)} = \frac{P_1P_2 \cdot P_1P_0^2}{4\left(P_1P_2 \cdot \frac{T_1P_0}{2}\right)} = \frac{6-\sqrt{2}}{4-\sqrt{2}} = \frac{11+\sqrt{2}}{7}.$$



Similarly, we let B_3 and B_4 be on the circle, so that

$$\angle B_3AC = \angle B_4AC = 30^\circ;$$

let P_3 and P_4 be the corresponding locations of P, and let T_2 be the point of intersection of P_3P_4 and AP_0.

The triangle AP_3P_4 is equilateral and we have:

$$\begin{aligned} AB_3 &= \sqrt{3}, \quad AP_3 = 1+\sqrt{3}, \\ AT_2 &= AP_3 \cdot \frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2}, \\ T_2P_0 &= 3 - \frac{3+\sqrt{3}}{2} = \frac{3-\sqrt{3}}{2}\,. \end{aligned}$$

From the Law of Cosines applied to ΔAP_0P_3, we obtain

$$\begin{aligned} P_0P_3^2 &= AP_0^2 + AP_3^2 - 2\cdot AP_0 \cdot AP_3 \cdot \cos 30^\circ \\ &= 9 + 4 + 2\sqrt{3} - 2\cdot \frac{\sqrt{3}}{2} \cdot 3 \cdot (1+\sqrt{3}) \\ &= 4 - \sqrt{3}\,. \end{aligned}$$

If R_2 is the circumradius of $\Delta P_0P_1P_2$, then it results that

$$R_2 = \frac{P_3P_4 \cdot P_3P_0^2}{4\left(P_3P_4 \cdot \frac{T_2P_0}{2}\right)} = \frac{4-\sqrt{3}}{3-\sqrt{3}} = \frac{9+\sqrt{3}}{6}\,.$$

Since $R_1 \neq R_2$, it follows that there is not a circle on which all points of S lie.

CC2. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that f is continuous and that $\int_0^1 f(a+tu)dt = 0$ for every point $a \in \mathbb{R}^2$ and every vector $u \in \mathbb{R}^2$ with $\|u\| = 1$. Show that f is constant.
(*Originally question # 4 from the 2012 University of Waterloo Big E Contest.*)

One incorrect solution was received. The problem remains open

CC3. All three sides of a right triangle are integers. Prove that the area of the triangle: is also an integer; is divisible by 3; and is even.
(*Originally question # 10 from the 2010 Manitoba Mathematical Competition.*)

Solved by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Daniel Văcaru, Piteşti, Romania; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Haley Williams, student, Auburn University at Montgomery, Montgomery, AL, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Apostolopoulos modified by the editor.

Suppose the legs of the triangle have lengths a and b, and the hypotenuse has length c. The area of the triangle is $\frac{ab}{2}$, so in order to prove the area is even and divisible by 3, it suffices to show $4|ab$ and $3|ab$.

In this light, let $d \in \{3,4\}$. Observe that for any integer x,

$$x^2 \equiv 0,1 \pmod{d} \tag{1}$$

Thus, if both a and b are not divisible by d, then $a^2 \equiv 1 \pmod{d}$ and $b^2 \equiv 1 \pmod{d}$, and hence

$$c^2 = a^2 + b^2 \equiv 2 \pmod{d},$$

which contradicts (1). Therefore, for $d \in \{3, 4\}$, $d|a$ or $d|b$, implying $d|ab$.

CC4. Suppose that $n \geq 3$. A sequence $a_1, a_2, a_3, \ldots, a_n$ of n integers, the first m of which are equal to -1 and the remaining $p = n - m$ of which are equal to 1, is called an MP sequence. Consider all of the products $a_i a_j a_k$ (with $i < j < k$) that can be calculated using the terms from an MP sequence $a_1, a_2, a_3, \ldots, a_n$. Determine the number of pairs (m, p) with $1 \leq m \leq p \leq 1000$ and $m + p \geq 3$ for which exactly half of these products are equal to 1.
(*Originally question B3 part b) from the 2011 Canadian Senior Mathematics Contest.*)

Solved together by Billy Jin, Waterloo Collegiate Institute, Waterloo, ON and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. One incorrect solution was received.

Call an MP sequence *balanced* if it satisfies the specified condition. For a given MP sequence let M and P denote the number of triples (a_i, a_j, a_k) where $i < j < k$ such that $a_i a_j a_k = -1$ or 1, respectively. Then the condition $M = P$ is successively equivalent to:

$$\binom{m}{3} + \binom{m}{1}\binom{p}{2} = \binom{p}{3} + \binom{p}{1}\binom{m}{2}$$

$$\frac{m(m-1)(m-2)}{6} + \frac{mp(p-1)}{2} = \frac{p(p-1)(p-2)}{6} + \frac{pm(m-1)}{2}$$

$$3mp(m-p) = m(m-1)(m-2) - p(p-1)(p-2)$$

$$3mp(m-p) = m^3 - p^3 - 3(m^2 - p^2) + 2(m-p) \qquad (1)$$

Hence by setting $m = p = c$ where $c \in \mathbb{N}$ and $1 < c \leq 1000$, we get 999 balanced MP sequences.

Suppose now that $m \neq p$. Then dividing both sides of (1) by $m - p$ yields

$$3mp = m^2 + mp + p^2 - 3(m+p) + 2$$

$$p^2 - (2m+3)p + (m^2 - 3m + 2) = 0 \qquad (2)$$

Since p is an integer, the discriminant, D, of (2) must be a perfect square.

Now, $D = (2m+3)^2 - 4(m^2 - 3m + 2) = 24m + 1$, so

$$24m + 1 = k^2 \text{ for some } k \in \mathbb{N} \text{ with } k \geq 5 \qquad (3)$$

Note that $k^2 \equiv 1 \pmod{8}$ (if and only if k is odd) and $k^2 \equiv 1 \pmod{3}$ (if and only if $k \equiv 1$ or $2 \pmod{3}$). Hence any odd $k \geq 5$ such that $k \not\equiv 0 \pmod{3}$ would satisfy (3) for some m which would yield an admissible pair (m, p) provided the condition $1 \leq m \leq p \leq 1000$ holds. To satisfy this condition we solve (2) and choose $p = \frac{2m+3+\sqrt{D}}{2} = m + \frac{k+3}{2} > m$.

Since $p \leq 1000$ we have $\frac{k^2-1}{24} + \frac{k+3}{2} \leq 1000$ which implies $k^2 + 12k + 35 \leq 24000$ or $(k+6)^2 \leq 24001$, so $k + 6 \leq 154$ or $k \leq 148$. Hence it remains to count the number, q, of odd integers k with $5 \leq k \leq 148$ which are not divisible by 3. Among the 72 odd integers in the set $\{5, 6, \cdots, 148\}$, exactly 24 of them are divisible by 3 since $k = 3 + 6d \leq 147$ implies $d \leq 24$. Hence, $q = 72 - 24 = 48$.

Therefore, the required number is $999 + 48 = 1047$.

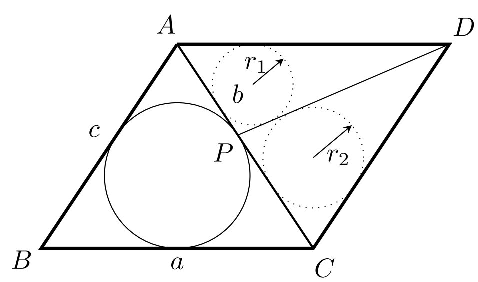
CC5. Let $ABCD$ be a parallelogram. We draw in the diagonal AC. A circle is drawn inside ΔABC tangent to all three sides and touches side AC at a point P. Draw in the line DP. A circle of radius r_1 is drawn inside ΔDAP tangent to all three sides and a circle of radius r_2 is drawn inside ΔDCP tangent to all three sides. Prove that

$$\frac{r_1}{r_2} = \frac{AP}{PC}.$$

(*Originally question C3 part b) from the 2012 Sun Life Financial Canadian Open Mathematics Challenge.*)

Solved by Daniel Văcaru, Piteşti, Romania; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

We denote $a = BC$, $b = AC$, $c = AB$.



It is well known that

$$AP = \frac{b+c-a}{2}, \quad \text{and} \quad PC = \frac{a+b-c}{2}.$$

Since

$$r_1 = \frac{\text{area}(\Delta APD)}{\frac{AP+PD+DA}{2}}, \quad r_2 = \frac{\text{area}(\Delta PCD)}{\frac{PC+CD+DP}{2}}, \quad \text{and} \quad \frac{\text{area}(\Delta APD)}{\text{area}(\Delta PCD)} = \frac{AP}{PC}.$$

Thus

$$\frac{r_1}{r_2} = \frac{AP(PC+CD+PD)}{PC(AP+PD+AD)} = \frac{AP}{PC} \times \frac{\frac{a+b-c}{2} + c + PD}{\frac{b+c-a}{2} + a + PD} = \frac{AP}{PC} \times \frac{\frac{a+b+c}{2} + PD}{\frac{a+b+c}{2} + PD},$$

and therefore $\frac{r_1}{r_2} = \frac{AP}{PC}$.