THE CONTEST CORNER

No. 11

Shawn Godin

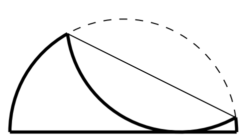
The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention LastName_FirstName_CCProblemNumber *(example Doe_Jane_CC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at* crux-contest@cms.math.ca. *Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page.*

To facilitate their consideration, solutions to the problems should be received by the editor by **1 May 2014**, *although solutions received after this date will also be considered until the time when a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the Solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC51. A semicircular piece of paper with radius 2 is creased and folded along a chord so that the arc is tangent to the diameter as shown in the diagram. If the contact point of the arc divides the diameter in the ratio 3 : 1, determine the length of the crease.



CC52. There are some marbles in a bowl. Alphonse, Beryl and Colleen each take turns removing one or two marbles from the bowl, with Alphonse going first, then Beryl, then Colleen, then Alphonse again, and so on. The player who takes the last marble from the bowl is the loser, and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can Beryl and Colleen work together and force Alphonse to lose?

CC53. Determine an infinite family of quadruples (a, b, c, d) of positive integers, each of which is a solution to $a^4 + b^5 + c^6 = d^7$.

CC54. Let k, l, m, n be positive integers such that $k + l + m \geq n$. Prove the following relation for binomial coefficients

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}.$$

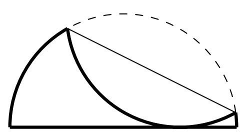
The summation in the left-hand side runs over all ordered partitions of n into three integers p, q, r such that $0 \leq p \leq k$, $0 \leq q \leq l$, $0 \leq r \leq m$.

CC55. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

. . . .

CC51. Un morceau de papier de forme semi-circulaire, de rayon 2, est plié le long d'une corde de manière que l'arc soit tangent au diamètre, comme dans la figure. Sachant que le point de contact de l'arc et du diamètre divise le diamètre dans un rapport de 3 : 1, déterminer la longueur du pli.



CC52. Un bol contient un nombre de billes. Alain, Bianca et Carla enlèvent tour à tour une ou deux billes du bol, en commençant par Alain, suivi de Bianca, puis de Carla, puis d'Alain et ainsi de suite. Le joueur qui enlève la dernière bille perd la joute, tandis que les deux autres sont gagnants. S'il y a N billes dans le bol au départ, pour quelles valeurs de N Bianca et Carla peuvent-elles travailler ensemble pour qu'Alain soit toujours perdant ?

CC53. Déterminer une famille infinie de quadruplets (a, b, c, d) d'entiers strictement positifs, chaque quadruplet étant une solution de l'équation $a^4 + b^5 + c^6 = d^7$.

CC54. Soit k, l, m et n des entiers strictement positifs tels que $k + l + m \geq n$. Démontrer la relation suivante qui comporte des coefficients binômiaux :

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}$$

La sommation du membre de gauche se fait sur toutes les partitions ordonné de n en trois entiers p, q, r de manière que $0 \leq p \leq k$, $0 \leq q \leq l$ et $0 \leq r \leq m$.

CC55. Soit α, β, γ les racines de l'équation $x^3 - x - 1 = 0$. Évaluer l'expression

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$