PROBLEM OF THE MONTH

No. 3

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*This column is dedicated to the memory of former **CRUX with MAYHEM** Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The "Problem of the Month" features a problem and solution that we know Jim would have liked.*

The integer nearest the real number x is often denoted by curly brackets, $\{x\}$. Unfortunately, in determining $\{x\}$, you never know whether x will need to be "rounded up" or to be "rounded down" until you have its numerical value in hand. This vagueness always makes me worry that a nearest integer function might behave erratically. Imagine my surprise and delight, then, in coming across the following engaging property.

For n a positive integer, prove that

$$\sum_{i=1}^{n^2+n} \{\sqrt{i}\} = 2(1^2 + 2^2 + 3^2 + \cdots + n^2).$$

The first thing we might notice is that, as n changes from n to $n+1$, the upper limit of the range of summation on the left side jumps from $n^2 + n$ to $(n+1)^2 + (n+1)$, thus introducing

$$[(n+1)^2 + (n+1)] - (n^2 + n) = 2(n+1)$$

new terms, namely

$$\{\sqrt{n^2+n+1}\}, \{\sqrt{n^2+n+2}\}, \ldots, \{\sqrt{(n+1)^2+(n+1)}\}.$$

On the other hand, the right side increases by $2(n+1)^2$, that is, by

$$2(n+1)(n+1).$$

Now, even the most optimistic among us might feel it is too much to hope that every one of these $2(n+1)$ new terms on the left side will turn out to be equal to $n+1$. Since it probably won't take us long to dispose of this outrageous notion, let's humor the dreamer who suggested it.

First of all, when is the integer nearest $\sqrt{i}$ equal to $n+1$? Since $\sqrt{i}$ is either an integer or is irrational, it is impossible for $\sqrt{i}$ to lie halfway between consecutive integers. Therefore

$$\{\sqrt{i}\} = n+1$$

if and only if

$$n+\frac{1}{2} < \sqrt{i} < n+\frac{3}{2}.$$

Squaring gives

$$n^2+n+\frac{1}{4} < i < n^2+3n+\frac{9}{4},$$

and since i is an integer, this is equivalent to

$$n^2+n+1 \leq i \leq n^2+3n+2,$$

that is, to

$$n^2+n+1 \leq i \leq (n+1)^2+(n+1).$$

These values of i are precisely its $2(n+1)$ new values on the left side, and so we can only applaud the boldness of our intrepid dreamer and rest in the knowledge that we have discovered an inductive solution to the problem.

This problem was proposed almost sixty years ago in the September issue of *Mathematics Magazine* by Joseph Lambek (McGill University) and Leo Moser (University of Alberta), and their concise solution by induction was duly published in the March-April issue in 1955, page 237.

An Exercise: While we are on the subject of "nearest integer", here's a cute little challenge from Polya and Szego's famous Problems and Theorems in Analysis - vol. 2, problem 5, page 111 (1976 Springer edition).

> *Let x be a real number that is not halfway between two consecutive integers. Express the integer nearest x, $\{x\}$, in terms of the "integer part" symbol* $[\cdot]$.

Everything is so easy and straightforward when the solution is laid before you; so, if you want to have some fun with this problem, take a few moments to figure it out for yourself before reading on.

Solution: A real number x is the sum of its integer part and its fractional part:

$$x = [x] + f, \text{ where } 0 \leq f < 1.$$

If $f < \frac{1}{2}$, then the nearest integer is just $[x]$, and if $f > \frac{1}{2}$, it is $[x]+1$.

Now, for an integer n, any non-zero fractional part of $n+x$ must be contained in the number x, and so

$$[n+x] = n + [x].$$

Therefore

$$[2x] = [2[x] + 2f] = 2[x] + [2f]$$

and, transposing $[x]$, we have

$$[2x] - [x] = [x] + [2f].$$

But $[x] + [2f]$ is just $[x]$ when $f < \frac{1}{2}$ and $[x] + 1$ when $f > \frac{1}{2}$. Hence $[x] + [2f]$ always yields the integer nearest x and it follows that

$$\{x\} = [2x] - [x].$$

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Unsolved Crux Problem

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Recently, Chris Fisher published a list of unsolved problems from *Crux* [2010 : 545, 547]. Below is one of these unsolved problems:

714*. [1982 : 48; 1983 : 58] *Proposed by Harry D. Ruderman, Hunter College, New York, NY, USA.*

Prove or disprove that for every pair (p, q) of non-negative integers there is a positive integer n such that

$$\frac{(2n-p)!}{n!(n+q)!}$$

is an integer.

(This problem was suggested by Problem 556 [1980 : 184; 1981 : 189, 241, 282] proposed by Paul Erdős.)