# THE OLYMPIAD CORNER

## No. 307

Nicolae Strungaru

*The solutions to the problems are due to the editor by* **1 March 2014**.

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.*

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**OC101**. Let $n, k$ be positive integers so that $1 < k < n-1$. Prove that the binomial coefficient $\binom{n}{k}$ is divisible by at least two distinct primes.

**OC102**. Let $\mathbb{N}$ denote the set of all nonnegative integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that both (1) and (2) are satisfied.

(1) $0 \leq f(x) \leq x^2$ for all $x \in \mathbb{N}$.

(2) $x - y$ divides $f(x) - f(y)$ for all $x, y \in \mathbb{N}$ with $x > y$.

**OC103**. Let $K$ and $L$ be the points on the semicircle with diameter $AB$. Denote the intersection of $AK$ and $BL$ as $T$ and let $N$ be the foot of the perpendicular from $T$ to $AB$. If $U$ is the intersection of the perpendicular bisector of $AB$ and $KL$ and $V$ is a point on $KL$ such that angles $UAV$ and $UBV$ are equal, then prove that $NV$ is perpendicular to $KL$.

**OC104**. Given a triangle $ABC$, let $D$ be the midpoint of the side $AC$ and let $M$ be the point on the segment $BD$ so that $BM : MD = 1 : 2$. The rays $AM$ and $CM$ intersect the sides $BC$ and $AB$ at $E$ and $F$ respectively. We know that $AM \perp CM$. Prove that the quadrilateral $AFED$ is cyclic if and only if the median from $A$ in $\Delta ABC$ meets the line $EF$ at a point situated on the circumcircle of $\Delta ABC$.

**OC105**. Let $n > 1$ be an integer, and let $k$ be the number of distinct prime divisors of $n$. Prove that there exists an integer $a$, $1 < a < \frac{n}{k} + 1$, such that $n \mid a^2 - a$.

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**OC101**. Soient $n$ et $k$ des entiers positifs tels que $1 < k < n-1$. Démontrer que le coefficient binomial $\binom{n}{k}$ est divisible par au moins deux nombres premiers distincts.

**OC102**. Soit $\mathbb{N}$ l'ensemble de tous les entiers non négatifs. Déterminer toutes les fonctions $f : \mathbb{N} \to \mathbb{N}$ telles que les deux conditions suivantes soient satisfaitent.

(1) $0 \leq f(x) \leq x^2$ pour tout $x \in \mathbb{N}$.

(2) $x - y$ divise $f(x) - f(y)$ pour tout $x, y \in \mathbb{N}$ tels que $x > y$.

**OC103**. Soient $K$ et $L$ des points sur le demi cercle de diamètre $AB$. Dénoter par $T$ le point d'intersection de $AK$ et $BL$ ; soit $N$ le pied de la perpendiculaire de $T$ vers $AB$. Soit $U$ le point d'intersection de la bissectrice perpendiculaire de $AB$ avec $KL$ ; soit $V$ un point sur $KL$ tel que les angles $UAV$ et $UBV$ soient égaux. Démontrer que $NV$ est perpendiculaire à $KL$.

**OC104**. Soit le triangle $ABC$. Soient aussi $D$ le mipoint du côté $AC$ et puis $M$ le point sur le segment $BD$ tel que $BM : MD = 1 : 2$. Les rayons $AM$ et $CM$ intersectent les côtés $BC$ et $AB$ aux points $E$ et $F$ respectivement. Nous savons que $AM \perp CM$. Démontrer que le quadrilatère $AFED$ est cyclique si et seulement si la médiane de $A$ dans $\Delta ABC$ rencontre la ligne $EF$ à un point situé sur le cercle circonscrit de $\Delta ABC$.

**OC105**. Soit $n > 1$ entier et soit $k$ le nombre de diviseurs premiers distincts de $n$. Démontrer qu'il existe un entier $a$, $1 < a < \frac{n}{k} + 1$, tel que $n \mid a^2 - a$.

# OLYMPIAD SOLUTIONS

**OC30**. Let $P$ be an interior point of a regular $n$-gon $A_1A_2\cdots A_n$. Each line $A_iP$ meets the $n$-gon at another point $B_i$. Prove that

$$\sum_{i=1}^{n} PA_i \geq \sum_{i=1}^{n} PB_i\,.$$

(*Originally question 8 from the 2008 China Western Mathematical Olympiad.*)

*Solved by George Apostolopoulos, Messolonghi, Greece.*

Let $m = \left\lceil \frac{n}{2} \right\rceil$, then, with the convention $A_{n+k} = A_k$ for all $k = 1, 2, ..., n$, the diagonals $A_kA_{k+m}$ are the longest diagonals in the polygon. Let $d$ denote the length of these diagonals. For each $A_i$ there exists some $j$ so that $B_i$ is a point on the edge $A_jA_{j+1}$, possibly one of the vertices $A_j$, $A_{j+1}$. Then

$$A_iB_i \leq \max\{A_iA_j, A_iA_{j+1}\} \leq d\,.$$

Thus,

$$PA_i + PB_i = A_iB_i \leq d\,.$$

As $A_iA_{i+m} = d$ we also get by the triangle inequality that

$$PA_i + PA_{i+m} \geq A_iA_{i+m} = d \geq PA_i + PB_i\,.$$

Hence,

$$PA_{i+m} \geq PB_i\,.$$

Adding these relations, we get the desired result.

**OC41**. Let $P$ be a point in the interior of a triangle $ABC$. Show that

$$\frac{PA}{BC} + \frac{PB}{AC} + \frac{PC}{AB} \geq \sqrt{3}\,.$$

(*Originally question 10 from the 2009 India IMO selection test.*)

*Similar solutions by Arkady Alt, San Jose, CA, USA ; Michel Bataille, Rouen, France ; Marian Dincă, Bucharest, Romania ; and David E. Manes, SUNY at Oneonta, Oneonta, NY, USA. We will give the solution of Dinca.*

We start by proving the Hayashi inequality:

$$\frac{PA \cdot PB}{CA \cdot CB} + \frac{PA \cdot PC}{BA \cdot BC} + \frac{PB \cdot PC}{AB \cdot AC} \geq 1\,.$$

To prove this inequality we proceed as follows. We view $A, B, C, P$ as points in the complex plane, and we denote by $a, b, c, z$ their complex coordinates.

Let

$$P(z) = (z-a)(z-b)(a-b) + (z-b)(z-c)(b-c) + (z-c)(z-a)(c-a)\,.$$

Then $P(z)$ is a polynomial of degree at most two, and it is easy to see that $P(a) = P(b) = P(c) = (a-b)(b-c)(c-a)$. Thus, $P(z)$ must be the constant polynomial $(a-b)(b-c)(c-a)$. Hence

$$(z-a)(z-b)(a-b)+(z-b)(z-c)(b-c)+(z-c)(z-a)(c-a) = (a-b)(b-c)(c-a)\,.$$

Then

$$\begin{aligned} AB \cdot AC \cdot BC &= |(a-b)(b-c)(c-a)| \\ &= |(z-a)(z-b)(a-b) + (z-b)(z-c)(b-c) \\ &\qquad\qquad + (z-c)(z-a)(c-a)| \\ &\leq |(z-a)(z-b)(a-b)| + |(z-b)(z-c)(b-c)| \\ &\qquad\qquad + |(z-c)(z-a)(c-a)| \\ &= PA \cdot PB \cdot AB + PB \cdot PC \cdot BC + PC \cdot PA \cdot CA\,. \end{aligned}$$

Dividing the inequality by $AB \cdot AC \cdot BC$ we get the Hayashi inequality.

Now, using the well known $(x+y+z)^2 \geq 3(xy+yz+zx)$ with $\frac{PA}{BC} = x$, $\frac{PB}{AC} = y$, $\frac{PC}{AB} = z$ we get

$$\left(\frac{PA}{BC} + \frac{PB}{AC} + \frac{PC}{AB}\right)^2 \geq 3\left(\frac{PA \cdot PB}{CA \cdot CB} + \frac{PA \cdot PC}{BA \cdot BC} + \frac{PB \cdot PC}{AB \cdot AC}\right) \geq 3\,.$$

**OC42**. Find the smallest $n$ for which $n!$ has at least 2010 different divisors.
(*Originally question 3 from the 2009-2010 Finish National Olympiad, Final round.*)

*Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA ; Mihaï-Ioan Stoënescu, Bischwiller, France ; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Manes.*

The smallest $n$ is 14.

Let $\tau(n)$ denote the number of divisors of $n$. As $\tau$ is a multiplicative function, with $\tau(p^\alpha) = \alpha + 1$ when $p$ is prime and $\alpha \geq 0$ is an integer, we get

$$\begin{aligned} \tau(13!) &= \tau(2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13) \\ &= (10+1)(5+1)(2+1)(1+1)(1+1)(1+1) = 1584\,, \\ \tau(14!) &= \tau(2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13) \\ &= (11+1)(5+1)(2+1)(2+1)(1+1)(1+1) = 2592\,. \end{aligned}$$

As $k!$ divides 13! for all $k \leq 13$, we know that $\tau(k!) \leq \tau(13!) = 1584$ for all $k \leq 13$, and this shows that $n = 14$ is the smallest number with the desired property.

**OC43**. Find all functions $f : \mathbb{R} \to \mathbb{R}$ verifying

$$f(x^3 + y^3) = xf(x^2) + yf(y^2)\,;\ \forall x, y \in \mathbb{R}\,.$$

(*Originally question 3 from the 2009 Romania National Olympiad, 10th grade.*)

*Solved by Michel Bataille, Rouen, France ; Oliver Geupel, Brühl, NRW, Germany ; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Geupel.*

The functions

$$f(x) = cx, \qquad c \in \mathbb{R}$$

satisfy the given functional equation, and we are going to prove that there are no other solutions.

Suppose that $f$ is a solution. Setting $y = 0$, we see that $f(x^3) = xf(x^2)$. Hence, $f(x^3 + y^3) = xf(x^2) + yf(y^2) = f(x^3) + f(y^3)$, which implies the identity $f(x+y) = f(x) + f(y)$. Also, $f(-x^3) = -xf(x^2) = -f(x^3)$, from which we obtain the identity $f(-x) = -f(x)$. We conclude

$$\begin{aligned} 0 &= f\left((x+1)^3\right) - (x+1)f\left((x+1)^2\right) + f\left((x-1)^3\right) - (x-1)f\left((x-1)^2\right) \\ &= f\left(x^3\right) + 3f\left(x^2\right) + 3f\left(x\right) + f(1) - (x+1)\left(f\left(x^2\right) + 2f(x) + f(1)\right) \\ &\qquad + f\left(x^3\right) - 3f\left(x^2\right) + 3f\left(x\right) - f(1) - (x-1)\left(f\left(x^2\right) - 2f(x) + f(1)\right) \\ &= 2(f(x) - f(1) \cdot x), \end{aligned}$$

that is, $f(x) = f(1) \cdot x$, which completes the proof.

**OC44**. In a scalene triangle $ABC$, we denote by $\alpha$ and $\beta$ the interior angles at $A$ and $B$. The bisectors of these angles meet the opposite sides of the triangle at points $D$ and $E$, respectively. Prove that the acute angle between the lines $D$ and $E$ does not exceed $\frac{|\alpha-\beta|}{3}$.
(*Originally question 1 from the 2009 Serbia Mathematical Olympiad, first day.*)

*Solved by Oliver Geupel, Brühl, NRW, Germany.*

Fixing the typo in the problem statement, we are going to prove that the acute angle between the lines $AB$ and $DE$ does not exceed $\frac{|\alpha-\beta|}{3}$. Moreover, we show that the inequality is strict.

Let $a$, $b$, and $c$ denote the lengths of the sides opposite to $A$, $B$, and $C$, respectively. Since the points $D$ and $E$ divide $BC$ and $CA$ in the ratios $c : b$ and $a : c$, respectively, we have

$$BD = \frac{ca}{b+c}, \quad CD = \frac{ba}{b+c}, \quad CE = \frac{ab}{c+a}, \quad AE = \frac{cb}{c+a}.$$

Let the lines $AB$ and $DE$ meet at point $F$. There is no loss of generality in assuming that $a > b$. Then, $A$ lies between $B$ and $F$. By Menelaus' theorem, it holds

$$\frac{AF}{AF+c} = \frac{AF}{BF} = \frac{CD}{BD} \cdot \frac{AE}{CE} = \frac{b}{a},$$

hence $AF = \frac{bc}{a-b}$ and $BF = \frac{ac}{a-b}$. [*Ed.* : $CF$ is the external bisector of angle $C$.]

Let $\delta$ denote the acute angle $AFE$. By the law of sines, in the triangles $AEF$ and $BDF$ it holds

$$\frac{\sin(\alpha-\delta)}{\sin\delta} = \frac{AF}{AE} = \frac{a+c}{a-b}, \qquad \frac{\sin(\beta+\delta)}{\sin\delta} = \frac{BF}{BD} = \frac{b+c}{a-b}.$$

Thus,

$$\sin\delta = \sin(\alpha-\delta) - \sin(\beta+\delta) = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta-2\delta}{2}.$$

Since $\sin\delta$ and $\cos\dfrac{\alpha+\beta}{2}$ are positive, we see that $\sin\dfrac{\alpha-\beta-2\delta}{2}$ is also positive, that is, $0 < \dfrac{\alpha-\beta-2\delta}{2} < \dfrac{\alpha+\beta}{2}$. We obtain

$$2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta-2\delta}{2} < 2\cos\frac{\alpha-\beta-2\delta}{2}\sin\frac{\alpha-\beta-2\delta}{2} = \sin(\alpha-\beta-2\delta).$$

Thus,

$$\sin\delta < \sin(\alpha-\beta-2\delta).$$

By the monotonicity of the sine function in the interval $[0,\ \pi/2]$ we deduce that $\delta < \alpha-\beta-2\delta$. The conclusion follows.

**OC45**. Let $a_1, a_2, a_3, ..., a_{15}$ be prime numbers forming an arithmetic progression with common difference $d > 0$. If $a_1 > 15$, prove that $d > 30,000$.
(*Originally question 3 from the 2009 Singapore Mathematical Olympiad, open section, round 2.*)

*Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Alex Song, Phillips Exeter Academy, NH, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Song and Wang.*

Let $p_n$ denote the $n$th prime. We prove the following general result:

*If $a_1, .., a_m$ are prime numbers in arithmetic progression, with common difference $d$ and if $a_1 > m > p_n$ then $d$ is divisible by $\prod_{k=1}^{n} p_k$.*

Indeed, assume by contradiction that $p_k \nmid d$ for some $1 \leq k \leq n$. Then $d$ is invertible modulo $p_k$, which implies that the equation

$$xd \equiv -a_1 \pmod{p_k}$$

has a solution $0 \leq r \leq p_k < m$. But then

$$a_r \equiv a_1 + rd \equiv 0 \pmod{p_k} ,$$

which implies $p_k \mid a_r$. As $a_r > a_1 > p_k$ , we get that $a_r$ is not prime, a contradiction.

In particular, in our problem $a_1 > 15 > 13 = p_6$, and hence $d$ is divisible by $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$, so $d > 30000$.