FOCUS ON ...

No. 4

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The Barycentric Equation of a Line

Introduction

Barycentric coordinates relative to a triangle ABC constitute a common and convenient tool in plane geometry. A point P has coordinates (x, y, z) (with $x+y+z \neq 0$) if P is the barycentre of A, B, C with respective masses x, y, z, that is, if $(x+y+z)\overrightarrow{MP} = x\overrightarrow{MA} + y\overrightarrow{MB} + z\overrightarrow{MC}$ for any point M. These coordinates are also called areal coordinates because x, y, z are proportional to the signed areas $[PBC], [PCA], [PAB]$, a nice geometric interpretation of x, y, z. In this context, the equation of a line is of the form $ux+vy+wz = 0$ for some real numbers u, v, w, not all zero, and this leads to systematic ways of solving problems of collinearity or concurrency. Stepping back, we would like to give a geometric look to the coefficients u, v, w and offer some applications.

Two simple results about u, v, w

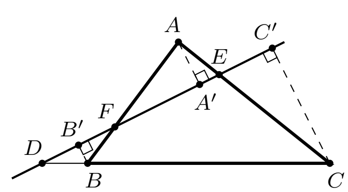
In this paragraph, we assume that u, v, w are not zero, leaving these special cases to the reader. Let ℓ be the line with equation $ux + vy + wz = 0$ and let ℓ intersect the sidelines BC, CA, AB at D, E, F respectively. Then we have

$$\frac{v}{w} = -\frac{BD}{DC}, \quad \frac{w}{u} = -\frac{CE}{EA}, \quad \frac{u}{v} = -\frac{AF}{FB} \tag{1}$$

and, if A', B', C' are the orthogonal projections of A, B, C onto ℓ,

$$\frac{AA'}{u} = \frac{BB'}{v} = \frac{CC'}{w} \tag{2}$$

where all distances are signed.



For example, if $(0, \beta, \gamma)$ are the coordinates of D, then $v\beta + w\gamma = 0$ and $\beta\overrightarrow{DB} + \gamma\overrightarrow{DC} = \overrightarrow{0}$, hence $w\overrightarrow{DB} = v\overrightarrow{DC}$. The first equality in (1) follows (alternatively, one can observe that $v[DCA] + w[DAB] = 0$ and $\frac{[DAB]}{[DCA]} = \frac{BD}{DC}$). As for (2), the homothety with centre D transforming B into C transforms B' into C', hence $\frac{DB}{DC} = \frac{BB'}{CC'}$. Similarly, $\frac{EC}{EA} = \frac{CC'}{AA'}$ and $\frac{FA}{FB} = \frac{AA'}{BB'}$ and (2) follows with the help of (1).

Another solution to a 2006 problem

The equalities (1) directly give Menelaus's relation for the transversal ℓ: indeed, $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -\frac{v}{w} \cdot \frac{w}{u} \cdot \frac{u}{v} = -1$. Not surprisingly, the barycentric equation of a line can be a shortcut avoiding the use of Menelaus's theorem. For example, consider Virgil Nicula's problem **3156** ([2006 : 305 ; 2007 : 312]):

Let Γ be the circumcircle of ΔABC. Let M be an interior point on the side AB, and let N be an interior point on the side AC. Let D be an intersection point of MN with Γ. Prove that

$$\left|\frac{MB}{MA}\cdot\frac{AC}{DB}-\frac{NC}{NA}\cdot\frac{AB}{DC}\right|=\frac{BC}{DA}. \tag{3}$$

In the featured solution, Peter Y. Woo applies Menelaus's theorem twice. Here is a shorter proof: From (1), the equation of the line MN can be written as $x = \frac{MB}{MA}\,y + \frac{NC}{NA}\,z$ (not signed distances). Since the signed areas $[DCA]$ and $[DAB]$ are of opposite signs, we obtain $[DBC] = \left|\frac{MB}{MA}[DCA] - \frac{NC}{NA}[DAB]\right|$ if areas are no longer signed. Because A, B, C, D are concylic, we have $\sin A = \sin\angle BDC$, $\sin B = \sin\angle CDA$, $\sin C = \sin\angle ADB$. It follows that $DB{\cdot}DC\sin A = \left|\frac{MB}{MA}\,DA\cdot DC\sin B - \frac{NC}{NA}\,DA\cdot DB\sin C\right|$ which, using the proportionality of BC, CA, AB and $\sin A$, $\sin B$, $\sin C$, easily leads to (3).

A property of the tangents to the circumcircle

To illustrate (2), we prove the following:

Let ℓ be a tangent to the circumcircle Γ of ΔABC and let $BC = a, CA = b, AB = c, d_a = d(A,\ell), d_b = d(B,\ell), d_c = d(C,\ell)$. Then, one of the numbers $a\sqrt{d_a}, b\sqrt{d_b}, c\sqrt{d_c}$ is the sum of the other two.

Proof. Since the equation of Γ is $a^2yz + b^2zx + c^2xy = 0$, the equation of ℓ is $x(b^2z_0 + c^2y_0) + y(c^2x_0 + a^2z_0) + z(a^2y_0 + b^2x_0) = 0$ where (x_0, y_0, z_0) are the coordinates of the point of tangency. Expressing that the coefficients of x, y, z are proportional to d_a, d_b, d_c (from (2)) and solving for x_0, y_0, z_0 give

$$x_0 : y_0 : z_0 = a^2(c^2d_c+b^2d_b-a^2d_a) : b^2(a^2d_a+c^2d_c-b^2d_b) : c^2(b^2d_b+a^2d_a-c^2d_c).$$

Now, $d_ax_0 + d_by_0 + d_cz_0 = 0$ leads to

$$a^4d_a^2 + b^4d_b^2 + c^4d_c^2 = 2a^2b^2d_ad_b + 2b^2c^2d_bd_c + 2c^2a^2d_cd_a,$$

that is,

$$(a\sqrt{d_a} + b\sqrt{d_b} + c\sqrt{d_c})(a\sqrt{d_a} + b\sqrt{d_b} - c\sqrt{d_c}) \times(b\sqrt{d_b} + c\sqrt{d_c} - a\sqrt{d_a})(c\sqrt{d_c} + a\sqrt{d_a} - b\sqrt{d_b}) = 0$$

and the result follows. A synthetic proof of this (perhaps new) property would be interesting.

An exercise

To conclude, we propose the following problem to the reader: Let E and F be points on the sides AC and AB, respectively. Show that $[PBC]$ is the geometric mean of $[PAB]$ and $[PCA]$ for some point P on the line segment EF if and only if $AE\cdot AF \geq 4\,CE\cdot BF$.