SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Due to a filing error, a number of readers' solutions got misplaced and were never acknowledged. The following solutions were received by the editor-in-chief: ARKADY ALT, San Jose, CA, USA(3624); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3626, 3634, 3635, 3636); MICHEL BATAILLE, Rouen, France (3624); VÁCLAV KONEČNÝ, Big Rapids, MI, USA (3542); PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (3639); EDMUND SWYLAN, Riga, Latvia (3620, 3634, 3635, 3638); and PETER Y. WOO, Biola University, La Mirada, CA, USA (3626, 3627, 3628, 3629, 3632, 3634, 3635, 3638). The editor apologizes sincerely for the oversight.

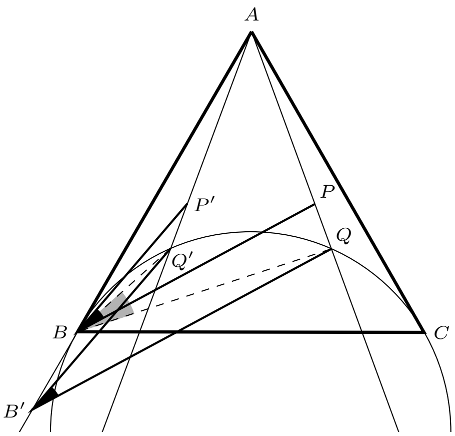
2597. [2000 : 499;2001 : 559] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

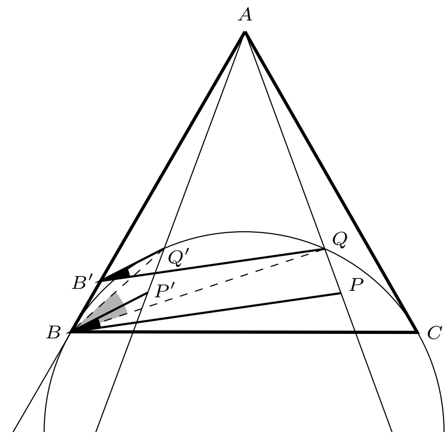
Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that $|\angle PBC - \angle PCB| \le \arcsin\left(2\sin\frac{|\angle PAB-\angle PAC|}{2}\right) - \frac{|\angle PAB-\angle PAC|}{2} \le |\angle PAB - \angle PAC|$. Show that the left inequality cannot be improved in the sense that there is a position Q of P on the ray AP giving an equality. (Thus the inequality in **2255** [1997: 300; 1998: 378-379; 1999: 113-114] is improved.)

Solution by Tomasz Cieśla, student, University of Warsaw, Poland.

When $\angle BAP = \angle PAC$ the given relations hold because the three quantities being compared are all zero; therefore, without loss of generality we shall assume that $\angle BAP > \angle PAC$ (and, consequently, $\angle PCB \ge \angle PBC$). Define ℓ to be that portion of the line AP in the interior of ΔABC. We first will prove that the position of P on ℓ that maximizes the quantity on the left is where $\angle BPC = \frac{2\pi}{3}$.

On ℓ choose point Q such that $\angle BQC = \frac{2\pi}{3}$. Denote by P' and Q' the reflections of P and Q in the bisector of angle BAC. Then we have $\angle PCB - \angle PBC = \angle PBP'$ and $\angle QCB - \angle QBC = \angle QBQ'$. Our claim is that $\angle QBQ' \ge \angle PBP'$ for all positions of P on ℓ. Consider homothety centered at A which sends P into Q. Then P' is sent to Q', and B is sent to some point B' lying on line AB. We have $\angle PBP' = \angle QB'Q'$. Since circle $(BQ'QC)$ is tangent to line AB at B (because $\angle BQC = \frac{2\pi}{3}$), we see that point B' lies outside the circle on the same side of line QQ' as B. This implies that $\angle QBQ' \ge \angle QB'Q' = \angle PBP'$ as claimed.

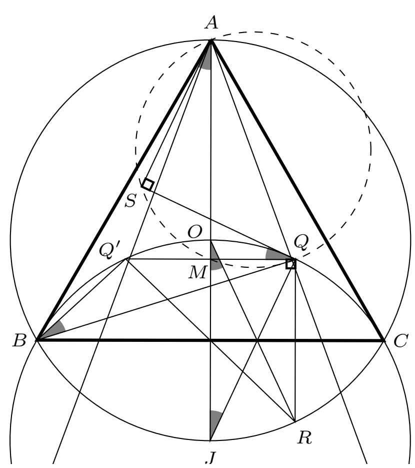




Next we will see that

$$\angle QBQ' = \arcsin\left(2\sin\frac{\angle QAB - \angle QAC}{2}\right) - \frac{\angle QAB - \angle QAC}{2}. \qquad (1)$$

Because the difference $\arcsin\left(2\sin\frac{|\angle PAB - \angle PAC|}{2}\right) - \frac{|\angle PAB - \angle PAC|}{2}$ is constant for all points P on ℓ, this will prove that the left inequality holds and, moreover, it cannot be improved.



Denote the circumcentre of ΔABC by O, and the reflections of Q and O in BC by R and J. Because triangle ABC is equilateral, points R, J lie on the circumcircle of ΔABC and J is the circumcenter of trapezoid $BCQQ'$. Note that O is midpoint of arc QQ' of circle $(BQQ'C)$. Angle chasing gives us

$$\angle Q'OQ = \pi - \angle QBQ' = \pi - \frac{1}{2}\angle QJQ' = \pi - \angle QJO = \pi - \angle JOR = \angle ROA.$$

In addition, $OQ' = OQ$ and $OR = OA$. Thus there exists a rotation about O which maps Q' to Q and R to A; denote by S the image of Q under this rotation.

Then $\angle QSA = \angle Q'QR = \frac{\pi}{2}$. Since O is the circumcenter of isosceles triangle $Q'QS$,

$$\angle SQQ' = 2\angle OQQ' = 2\angle OBQ' = \angle QBQ'. \qquad (2)$$

Let M be midpoint of QQ'. Points Q, M, S, A lie on the circle with diameter QA, because $\angle QMA = \frac{\pi}{2} = \angle QSA$. Thus

$$\angle SQQ' = \angle SQM = \angle SAM. \qquad (3)$$

Observe that $\sin\angle SAQ = \frac{SQ}{AQ} = \frac{Q'Q}{AQ} = \frac{2MQ}{AQ} = 2\sin\angle MAQ = 2\sin\angle OAQ$. From that we get

$$\angle SAQ = \arcsin(2\sin\angle OAQ). \qquad (4)$$

From $\angle OAQ = \frac{\angle QAB - \angle QAC}{2}$ and equations (2) through (4),

$$\angle QBQ' = \angle SQQ' = \angle SAM = \angle SAQ - \angle OAQ,$$

which is equation (1), as claimed.

For the inequality on the right, simply note that we have proved that the middle difference is the maximum of $|\angle PCB - \angle PBC|$ over all points $P \in \ell$, while Problem 2255 established that this difference is at most $|\angle PAB - \angle PAC|$. This observation concludes the proof.

Also solved by the proposer; no solution was published before now.

For an alternative proof of the right inequality, let $x = |\angle PAB - \angle PAC|$, $0 \le x < \frac{\pi}{3}$. *The inequality to prove reduces to* $\arcsin\left(2\sin\frac{x}{2}\right) \le \frac{3x}{2}$, *for* $0 \le x < \frac{\pi}{3}$, *which is an elementary exercise. It is interesting to note that according to the solution of Problem 2255, the inequality there, namely* $|\angle PAB - \angle PAC| \ge |\angle PCB - \angle PBC|$, *holds for all isosceles triangles* ABC *for which* $\angle A \ge \frac{\pi}{3}$ *(and* $\angle B = \angle C \le \frac{\pi}{3}$*), while the inequality fails for some positions of* P *in isosceles triangles with* $\angle A < \frac{\pi}{3}$. *Note that* $\arcsin\left(2\sin\frac{x}{2}\right)$ *is no longer real for* $x > \frac{\pi}{3}$, *so that there are positions of* P *for which the right inequality of the present problem fails for isosceles triangles with* $\angle A > \frac{\pi}{3}$.

3641. [2011 : 234, 237] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $0 \le x_1, x_2, \ldots, x_n < \pi/2$ be real numbers. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(1 - \left(\frac{1}{n}\sum_{k=1}^{n}\sin(x_k)\right)^2\right)^{1/2} \ge 1\,.$$

I. Composite of similar solutions by Arkady Alt, San Jose, CA, USA; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Let $f(x) = \sec x$, $g(x) = \sin x$ and set $\bar{x} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k$. Since $f''(x) = \dfrac{1+\sin^2 x}{\cos^3 x} > 0$ and $g''(x) = -\sin x < 0$ for $0 < x < \dfrac{\pi}{2}$, f is convex and g is concave on the interval $(0, 1)$.

Hence Jensen's Inequality ensures that

$$\frac{1}{n}\sum_{k=1}^{n}\sec x_k \geq \sec(\bar{x}) \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n}\sin x_k \leq \sin(\bar{x}).$$

Therefore we have

$$\begin{aligned}\left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(1-\left(\frac{1}{n}\sum_{k=1}^{n}\sin(x_k)\right)^2\right)^{1/2} &\geq \sec(\bar{x})(1-\sin^2(\bar{x}))^{1/2}\\ &= \sec(\bar{x})\cos(\bar{x}) = 1.\end{aligned}$$

II. Composite of virtually identical solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Salem Malikić, student, Simon Fraser University, Burnaby, BC.

By Cauchy-Schwarz Inequality we have

$$n\left(\sum_{k=1}^{n}\sin^2(x_k)\right) = \left(\sum_{k=1}^{n}1^2\right)\left(\sum_{k=1}^{n}\sin^2(x_k)\right) \geq \left(\sum_{k=1}^{n}\sin(x_k)\right)^2$$

so

$$\left(\frac{1}{n}\sum_{k=1}^{n}\sin(x_k)\right)^2 \leq \frac{1}{n}\sum_{k=1}^{n}\sin^2(x_k).$$

Hence,

$$\begin{aligned}&\left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(1-\left(\frac{1}{n}\sum_{k=1}^{n}\sin(x_k)\right)^2\right)^{1/2}\\ &\qquad\geq \left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(1-\frac{1}{n}\sum_{k=1}^{n}\sin^2(x_k)\right)^{1/2}\\ &\qquad= \left(\frac{1}{n}\sum_{k=1}^{n}\sec(x_k)\right)\left(\frac{1}{n}\sum_{k=1}^{n}(1-\sin^2(x_k))\right)^{1/2}\\ &\qquad= \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\cos(x_k)}\right)\left(\frac{1}{n}\sum_{k=1}^{n}\cos^2(x_k)\right)^{1/2}\\ &\qquad\geq \left(\prod_{k=1}^{n}\frac{1}{\cos(x_k)}\right)^{1/n}\left(\left(\prod_{k=1}^{n}\cos^2(x_k)\right)^{1/n}\right)^{1/2} = 1\end{aligned}$$

by the AM-GM Inequality.

Clearly, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3642. [2011 : 235, 237] *Proposed by Michel Bataille, Rouen, France.*

Evaluate

$$\lim_{n\to\infty} \frac{\int_0^1 (2x^2-5x-1)^n dx}{\int_0^1 (x^2-4x-1)^n dx}\,.$$

Solution by Paul Bracken, University of Texas, Edinburg, TX, USA; modified by the editor.

Write the limit as $\lim\limits_{n\to\infty} I_1(n)/I_2(n)$, where $I_1(n) = \int_0^1 (1+5x-2x^2)^n dx$ and $I_2(n) = \int_0^1 (1+4x-x^2)^n dx$, and let $\phi(x) = \ln(1+5x-2x^2)$. Integration by parts gives

$$\begin{aligned} I_1(n) &= \int_0^1 e^{n\phi(t)}\,dt \;=\; \frac{1}{n}\int_0^1 \frac{1}{\phi'(t)}\frac{d}{dt}[e^{n\phi(t)}]\,dt \\ &= e^{n\phi(t)}\Big|_0^1 - \frac{1}{n}\int_0^1 \frac{d}{dt}\left[\frac{1}{\phi'(t)}\right] e^{n\phi(t)}\,dt \\ &= \frac{4^{n+1}}{n} - \frac{1}{5n} + \frac{1}{n}\int_0^1 \frac{d}{dt}\left[\frac{1}{\phi'(t)}\right] e^{n\phi(t)}\,dt\,. \end{aligned}$$

The function $d/dt[1/\phi'(t)] = 1/2 + (33/2)(5-4t)^{-2}$ increases on $[0,1]$, taking the value $29/25$ at $t=0$ and the value 17 at $t=1$. It follows that

$$\frac{29}{25}\int_0^1 e^{n\phi(t)}\,dt \;\le\; \int_0^1 \frac{d}{dt}\left[\frac{1}{\phi'(t)}\right] e^{n\phi(t)}\,dt \;\le\; 17\int_0^1 e^{n\phi(t)}\,dt,$$

and hence there is a constant C_1 with $29/25 \le C_1 \le 17$ and such that

$$\int_0^1 \frac{d}{dt}\left[\frac{1}{\phi'(t)}\right] e^{n\phi(t)}\,dt \;=\; C_1\int_0^1 e^{n\phi(t)}\,dt\,.$$

Thus,

$$I_1(n) \;=\; \frac{4^{n+1}}{n} - \frac{1}{5n} + \left(\frac{C_1}{n}\right) I_1(n)$$

and solving for $I_1(n)$ gives

$$I_1(n) \;=\; \left(1-\frac{C_1}{n}\right)^{-1}\cdot\frac{1}{n}\cdot\left(4^{n+1}-\frac{1}{5}\right)\,.$$

By similar calculations there is a constant C_2 with $9/8 \le C_2 \le 3$ such that

$$I_2(n) \;=\; \left(1-\frac{C_2}{n}\right)^{-1}\cdot\frac{1}{n}\cdot\left(2\cdot 4^{n}-\frac{1}{4}\right)\,.$$

Finally,

$$\lim_{n\to\infty}\frac{I_1(n)}{I_2(n)} \;=\; \lim_{n\to\infty}\frac{\left(1-\frac{C_2}{n}\right)\left(2-\frac{1}{10\cdot 4^n}\right)}{\left(1-\frac{C_1}{n}\right)\left(1-\frac{1}{2\cdot 4^{n+1}}\right)} \;=\; 2\,.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer. One incorrect solution and one incomplete solution were received.

3643. [2011 : 235, 238] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let u and v be positive real numbers. Prove that

$$\frac{1}{8}\left(17-\frac{2uv}{u^2+v^2}\right) \ \le\ \sqrt[3]{\frac{u}{v}}+\sqrt[3]{\frac{v}{u}} \ \le\ \sqrt{(u+v)\left(\frac{1}{u}+\frac{1}{v}\right)}$$

For each inequality, determine when equality holds.

Editor's note: Perfetti pointed out that the very same problem (by the same proposer) has appeared in Mathematics Magazine *(Vol. 82, No. 3, 2009) and a solution was published in Vol. 83, No. 3, pp. 229 – 230. However, we decide to publish a different solution which is completely elementary.*

Solution by Titu Zvonaru, Comăneşti, Romania.

Let $x=\sqrt[3]{\frac{u}{v}}$. Then $x>0$ and $x^3=\frac{u}{v}$.

The left inequality is equivalent, in succession, to

$$\begin{aligned}
\frac{1}{8}\left(17-\frac{2x^3}{x^6+1}\right) &\le x+\frac{1}{x}\\
\frac{17x^6-2x^3+17}{x^6+1} &\le \frac{8x^2+8}{x}\\
8x^8-17x^7+8x^6+2x^4+8x^2-17x+8 &\ge 0\\
(x-1)^2(8x^6-x^5-2x^4-3x^3-2x^2-x+8) &\ge 0\\
(x-1)^2((8x^6-x^5-2x^4-10x^3-2x^2-x+8)+7x^3) &\ge 0\\
(x-1)^2((x-1)^2(8x^4+15x^3+20x^2+15x+8)+7x^3) &\ge 0
\end{aligned}$$

which is clearly true.

To establish the right inequality note that $(u+v)\left(\frac{1}{u}+\frac{1}{v}\right)=x^3+\frac{1}{x^3}+2$ and

$$\begin{aligned}
x+\frac{1}{x}\le\sqrt{x^3+\frac{1}{x^3}+2} &\Leftrightarrow x^2+\frac{1}{x^2}+2\le x^3+\frac{1}{x^3}+2\\
&\Leftrightarrow x^6-x^5-x+1\ge 0 \Leftrightarrow (x-1)(x^5-1)\ge 0\\
&\Leftrightarrow (x-1)^2(x^4+x^3+x^2+x+1)\ge 0
\end{aligned}$$

which clearly holds. Note that equality holds in either inequality if and only if $x=1$; that is, if and only if $u=v$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3644. [2011 : 235, 238] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

We trisect the sides AB and AC of triangle ABC with the points D, E and F, G respectively such that $AE = ED = DB$ and $AF = FG = GC$. The line BF intersects CD, CE in the points K, L respectively, while BG intersects CD, CE in N, M respectively.

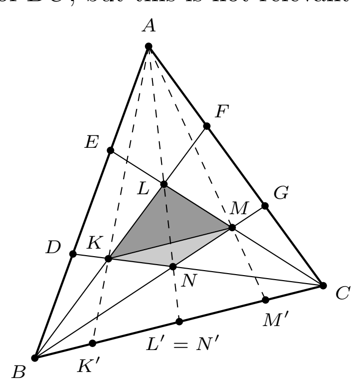
Prove that:

(a) KM is parallel to BC;

(b) $\text{Area}(KLM) = \frac{5}{7}\text{Area}(KLMN)$.

Composite of complementary solutions by Edmund Swylan, Riga, Latvia; and Titu Zvonaru, Comăneşti, Romania.

Let K', L', M', and N' be the points where the side BC meets the lines joining A to K, L, M, and N, respectively. (One can easily show that, in fact, $L' = N'$ is the midpoint of BC, but this is not relevant to our work here.)



By applying Van Aubel's theorem (see, for example, F. G.-M., *Exercices de géométrie—comprenant l'exposé des méthodes géométriques et 2000 questions résolues*, quatrième édition, J. De Gigord, Paris (1907), paragraph 1242j, page 542) four times, we have

$$\frac{AK}{KK'} = \frac{AD}{DB} + \frac{AF}{FC} = 2 + \frac{1}{2}, \quad \text{or} \quad \frac{KK'}{AK'} = \frac{2}{7};$$
$$\frac{AL}{LL'} = \frac{AE}{EB} + \frac{AF}{FC} = \frac{1}{2} + \frac{1}{2}, \quad \text{or} \quad \frac{LL'}{AL'} = \frac{1}{2};$$
$$\frac{AM}{MM'} = \frac{AE}{EB} + \frac{AG}{GC} = \frac{1}{2} + 2, \quad \text{or} \quad \frac{MM'}{AM'} = \frac{2}{7};$$
$$\frac{AN}{NN'} = \frac{AD}{DB} + \frac{AG}{GC} = 2 + 2, \quad \text{or} \quad \frac{NN'}{AN'} = \frac{1}{5}.$$

From the first and third of these equations we get $\frac{AK}{KK'} = \frac{AM}{MM'}$, whence $KM||K'M'$, and part (a) follows immediately. For part (b) we assume without loss of generality that the altitude from A to BC has length 1. The above equations then imply that the line segments perpendicular to BC from K, L, M, N equal $\frac{2}{7}, \frac{1}{2}, \frac{2}{7}, \frac{1}{5}$, respectively. Thus

$$\frac{\text{Area}(KLM)}{\text{Area}(KMN)} = \frac{\frac{1}{2} - \frac{2}{7}}{\frac{2}{7} - \frac{1}{5}} = \frac{5}{2},$$

and therefore

$$\frac{\text{Area}(KLM)}{\text{Area}(KLMN)} = \frac{5}{7}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA(2 solutions); and MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL.

3645. [2011 : 235, 238] *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be positive numbers such that $a^2 + b^2 + c^2 + 2abc = 1$. Prove that

$$\sum_{\text{cyclic}} \sqrt{a\left(\frac{1}{b} - b\right)\left(\frac{1}{c} - c\right)} \; > \; 2\,.$$

I. Solution by Arkady Alt, San Jose, CA, USA.

Observe that $0 < a, b, c < 1$ so that $abc \neq 1$. The inequality is equivalent to

$$a\sqrt{(1-b^2)(1-c^2)} + b\sqrt{(1-c^2)(1-a^2)} + c\sqrt{(1-a^2)(1-b^2)} > 2\sqrt{abc}. \qquad (1)$$

Since

$$(1-b^2)(1-c^2) = 1 - b^2 - c^2 + b^2c^2 = a^2 + 2abc + (bc)^2 = (a+bc)^2,$$

and, similarly, $(1-c^2)(1-a^2) = (b+ca)^2$ and $(1-a^2)(1-b^2) = (c+ab)^2$, the left side is equal to

$$\begin{aligned} a(a+bc) + b(b+ca) + c(c+ab) &= a^2 + b^2 + c^2 + 3abc \\ &= 1 + abc > 2\sqrt{abc} \end{aligned}$$

by the Arithmetic-Geometric Means Inequality.

II. Solution using ideas from Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, NRW, Germany; Salem Malikić, student, Simon Fraser University, Burnaby, BC; and Albert Stadler, Herrliberg, Switzerland.

We can select acute angles A, B, C for which $a = \cos A$, $b = \cos B$, $c = \cos C$. Then

$$\begin{aligned}\cos(A+B+C) &= \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C \\ &\qquad\qquad - \sin A\sin B\cos C \\ &= abc - a(a+bc) - b(b+ca) - c(c+ab) = -1,\end{aligned}$$

so that $A + B + C = \pi$. Therefore

$$\begin{aligned}2a\sqrt{(1-b^2)(1-c^2)} &= \cos A[\cos(B-C) - \cos(B+C)] \\ &= -\cos(B+C)\cos(B-C) + \cos^2 A \\ &= -\frac{1}{2}(\cos 2B + \cos 2C) + \cos^2 A \\ &= -\cos^2 B - \cos^2 C + 1 + \cos^2 A \\ &= 1 + a^2 - b^2 - c^2,\end{aligned}$$

with similar equations for the other two terms of the left side of (1) in the first solution. Therefore, the left side of (1) is equal to

$$\frac{1}{2}(3 - a^2 - b^2 - c^2) = 1 + abc > 2\sqrt{abc}.$$

Also solved by the proposer.

3646. [2011 : 235, 238] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $\alpha \geq 0$ and let β be a positive number. Find the limit

$$L(\alpha,\beta) \;=\; \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right)^k - n\right).$$

Solution by Anastasios Kotrononis, Athens, Greece; modified slightly by the editor.

We will prove that

$$L(\alpha,\beta) = \begin{cases} \infty & \text{if} \quad \beta - \alpha < 2 \\ 0 & \text{if} \quad \beta - \alpha > 2 \\ \dfrac{1}{\beta} & \text{if} \quad \beta - \alpha = 2 \end{cases}$$

by considering three cases separately.

Case (i) $\beta - \alpha < 2$. By Bernoulli's Inequality, we have

$$\left(1+\frac{k^\alpha}{n^\beta}\right)^k \geq 1 + \frac{k^{\alpha+1}}{n^\beta}$$

so

$$\sum_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right)^k - n \geq \sum_{k=1}^{n}\left(1+\frac{k^{\alpha+1}}{n^\beta}\right) - n = \sum_{k=1}^{n}\frac{k^{\alpha+1}}{n^\beta}$$
$$= n^{\alpha-\beta+2}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{\alpha+1}\right). \quad (1)$$

Note that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{\alpha+1} = \int_0^1 x^{\alpha+1}dx = \frac{1}{\alpha+2}. \quad (2)$$

Since $\alpha-\beta+2>0$ and $\alpha+2>0$, we conclude from (1) and (2) that $L(\alpha,\beta)=\infty$.

Case (ii) $\beta-\alpha>2$. Note first that for all $k=1,2,\ldots,n$ and $i=0,1$, we have

$$0<\frac{k^{\alpha+i}}{n^\beta}\leq\frac{1}{n^{\beta-\alpha-i}}<\frac{1}{n^{2-i}}.$$

In particular,

$$0<\frac{k^\alpha}{n^\beta}<\frac{1}{n^2} \quad \text{and} \quad 0<\frac{k^{\alpha+1}}{n^\beta}<\frac{1}{n}. \quad (3)$$

It is well known that as $x\to 0^+$ we have

$$\ln(1+x)=x+O(x^2) \quad (4)$$

and

$$e^x=1+x+O(x^2). \quad (5)$$

Using (3), (4) and (5), we have, as $n\to\infty$, that

$$\left(1+\frac{k^\alpha}{n^\beta}\right)^k = \exp\left(k\ln\left(1+\frac{k^\alpha}{n^\beta}\right)\right)$$
$$= \exp\left(k\left(\frac{k^\alpha}{n^\beta}+O\left(n^{2(\alpha-\beta)}\right)\right)\right)$$
$$= \exp\left(\frac{k^{\alpha+1}}{n^\beta}+O\left(n^{2(\alpha-\beta)+1}\right)\right)$$
$$= 1+\frac{k^{\alpha+1}}{n^\beta}+O\left(n^{2(\alpha-\beta)+2}\right)$$

so

$$\sum_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right)^k - n = \sum_{k=1}^{n}\frac{k^{\alpha+1}}{n^\beta}+O\left(n^{2(\alpha-\beta)+3}\right)$$
$$= \frac{1}{n^{\beta-\alpha-2}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{\alpha+1}\right)+O\left(n^{2(\alpha-\beta)+3}\right). \quad (6)$$

Since $\beta-\alpha-2>0$ and $2(\alpha-\beta)+3<-1$, it follows from (2) and (6) that $L(\alpha,\beta)=0$.

Case (iii) $\beta - \alpha = 2$. We proceed as in case (ii). Since $\beta - \alpha - 2 = 0$, it follows from (2) and (6) again that $L(\alpha, \beta) = \frac{1}{\alpha+2} = \frac{1}{\beta}$

This completes our proof.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and the proposer.

3647. [2011 : 236, 238] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Show that in triangle ABC with exradii r_a, r_b and r_c,

$$\sum_{\text{cyclic}} \frac{(r_a + r_b)(r_b + r_c)}{ac} \ \geq \ 9\,,$$

where $AB = c$, $BC = a$, and $CA = b$.

Similar solutions by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India; and Kee-Wai Lau, Hong Kong, China.

We know that $r_a = \frac{\Delta}{s-a}, r_b = \frac{\Delta}{s-b}$, and $r_c = \frac{\Delta}{s-c}$, where $s = \frac{a+b+c}{2}$ is the semiperimeter of ABC and $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ is its area. It follows that

$$r_a + r_b = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} = \frac{c\Delta}{(s-a)(s-b)}$$

and

$$r_b + r_c = \frac{\Delta}{s-b} + \frac{\Delta}{s-c} = \frac{a\Delta}{(s-b)(s-c)},$$

whence

$$\frac{(r_a + r_b)(r_b + r_c)}{ac} = \frac{\Delta^2}{(s-a)(s-b)^2(s-c)} = \frac{s}{s-b}.$$

Similarly,

$$\frac{(r_b + r_c)(r_c + r_a)}{ba} = \frac{s}{s-c} \quad \text{and} \quad \frac{(r_c + r_a)(r_a + r_b)}{cb} = \frac{s}{s-a}.$$

These last three equations give us

$$\sum_{\text{cyclic}} \frac{(r_a + r_b)(r_b + r_c)}{ac} = \frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c}. \tag{1}$$

But by the AM-HM inequality,

$$\frac{1}{3}\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) \geq \frac{3}{s-a+s-b+s-c} = \frac{3}{s},$$

which, when multiplied by $3s$, yields

$$\frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c} \geq 9. \tag{2}$$

The desired inequality follows from (1) and (2). Equality holds if and only if the triangle is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; EDMUND SWYLAN, Riga, Latvia; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3648. [2011 : 236, 239] *Proposed by Michel Bataille, Rouen, France.*

Find all real numbers x, y, z such that $xyz = 1$ and $x^3 + y^3 + z^3 = \frac{S(S-4)}{4}$ where $S = \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z}$.

I. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

We assume the equation

$$x^3 + y^3 + z^3 = \frac{S(S-4)}{4}(xyz),$$

without the restriction on xyz. This is equivalent to

$$\sum (x^4y^2 + x^3y^3 + x^2y^2z^2) = \sum (x^4yz + 2x^3y^2z),$$

where both sums, taken over the six permutations of the variables, are symmetric.

By the Arithmetic-Geometric Means Inequality,

$$\begin{aligned}\sum x^4y^2 &= x^4(y^2+z^2) + y^4(z^2+x^2) + z^4(x^2+y^2) \\ &\geq 2x^4yz + 2y^4zx + 2z^4xy = \sum x^4yz,\end{aligned}$$

with equality if and only if $x = y = z$.

Recall Schur's Inequality that, for $a, b, c \geq 0$,

$$\begin{aligned}&(a^3+b^3+c^3) + 3abc - (a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) \\ &\qquad = a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0.\end{aligned}$$

Setting $(a, b, c) = (yz, zx, xy)$, we obtain that

$$\sum (x^3y^3 + x^2y^2z^2) \geq 2\sum x^3y^2z,$$

where again each sum is symmetric with six terms.

Therefore

$$\sum (x^4y^2 + x^3y^3 + x^2y^2z^2) \geq \sum (x^4yz + 2x^3y^2z).$$

Since we are assuming that equality holds and that $xyz = 1$, the given equation is satisfied if and only if $x = y = z = 1$.

II. Solution by the proposer.

The given conditions imply that $S = xy^2 + yx^2 + yz^2 + zy^2 + zx^2 + xz^2$. Without loss of generality, let x be the maximum of x, y, z. Define

$$T = 4(xy^2 + yx^2 + yz^2)(zx^2 + xz^2 + zy^2).$$

Then

$$T = S^2 - (y-z)^2(x^2 + xy + xz - yz)^2$$

and also

$$T = 4xyz(x^3 + y^3 + z^3 + S) + 4y^2z^2(x-y)(x-z),$$

whence

$$S^2 = 4(x^3 + y^3 + z^3 + S) + 4y^2z^2(x-y)(x-z) + (y-z)^2(x^2 + xy + xz - yz)^2.$$

Since, by hypothesis, $4(x^3 + y^3 + z^3 + S) = S^2$, we deduce that

$$4y^2z^2(x-y)(x-z) + (y-z)^2(x^2 + xy + xz - yz)^2 = 0.$$

Since $x \geq y, z$, both terms on the left are nonnegative and therefore must vanish. If, say, $x = y$, then $x^2 + xy + xz - yz = 2x^2 \neq 0$, so that $y = z$. Since $xyz = 1$, we must have that $x = y = z = 1$.

No other solutions were received.

3649. [2011 : 236, 239] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, and c be three positive real numbers and let

$$k = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Prove that

$$(a^3 + b^3 + c^3)\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) \geq \frac{k^3 - 15k^2 + 63k - 45}{4},$$

and equality holds if and only if $(a, b, c) = \left(\frac{k - 5 \pm \sqrt{k^2 - 10k + 9}}{4}, 1, 1\right)$ or any of its permutations.

Solution by Oliver Geupel, Brühl, NRW, Germany.

All sums shall be cyclic. Let $x = \sum \frac{a}{b}$, $y = \sum \frac{b}{a}$, $m = \sum \frac{a^2}{bc}$, and $n = \sum \frac{bc}{a^2}$. We have $x + y = k - 3$, hence $4xy \leq (k-3)^2$. Using the relations

$$\sum \frac{a^3}{b^3} = x^3 - 3(m+n) - 6,$$
$$\sum \frac{b^3}{a^3} = y^3 - 3(m+n) - 6,$$

and $m + n = xy - 3$, we deduce that

$$\begin{aligned}(a^3+b^3+c^3)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) &= x^3+y^3+9-6xy\\ &= (x+y)^3-3(x+y)xy+9-6xy\\ &= (k-3)^3+9-3(k-1)xy\\ &\ge (k-3)^3+9-3(k-1)\cdot\frac{(k-3)^2}{4}\\ &= \frac{k^3-15k^2+63k-45}{4}\,.\end{aligned}$$

This proves the inequality.

Equality holds if and only if $0 = x - y = (a-b)(b-c)(c-a)/abc$, that is, if two of a, b, c coincide. Without loss of generality, suppose that $b = c$. The equality is then equivalent to $(k-3)/2 = x = 1 + a/b + b/a$. However, this holds if and only if $p = a/b$ is a root of the quadratic $p^2 - \left(\frac{k-5}{2}\right)p + 1$. The condition for equality (up to permutation) therefore needs to be corrected to

$$(a,b,c) = \left(\lambda\cdot\frac{k-5\pm\sqrt{k^2-10k+9}}{4}, \lambda, \lambda\right)\,,$$

where $\lambda > 0$.

Also solved by the proposers. Our featured solver said his solution was similar to and inspired by the solution to problem 75 in Secrets in Inequalities (Vol. I) *by Pham Kim Hung, GIL Publishing House, Zalău, 2007, pp. 214-215.*

3650. [2011 : 318, 320] *Replacement. Proposed by Michel Bataille, Rouen, France.*

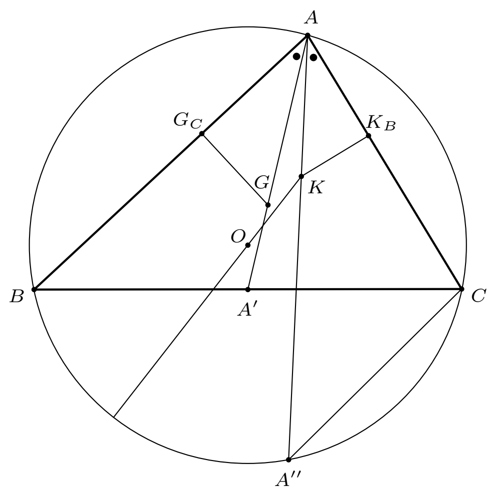
Let ABC be a triangle and R, O, G and K its circumradius, circumcentre, centroid and Lemoine point, respectively. Prove that

$$BC\cdot\frac{KA}{GA} = CA\cdot\frac{KB}{GB} = AB\cdot\frac{KC}{GC} = \sqrt{3(R^2-OK^2)}.$$

Recall that a symmedian of a triangle is the reflection of the median from a vertex in the angle bisector of the same vertex. The Lemoine point of a triangle is the point of intersection of the three symmedians.

Solution by Edmund Swylan, Riga, Latvia.

Let the length of BC be denoted by a and its midpoint by A'. Let the symmedian to BC meet the circumcircle again at A'' and, finally, let G_c be the foot of the perpendicular from G to AB and K_b be the foot of the perpendicular from K to AC.



Using, in turn, the similar right triangles AG_cG and AK_bK, the fact that GG_c equals a third of the altitude from C, and the sine law, we deduce that

$$\frac{KA}{GA} = \frac{KK_b}{GG_c} = \frac{KK_b}{\frac{1}{3}a\sin B} = \frac{KK_b \cdot 2R \cdot 3}{ab},$$

with analogous formulas for $\frac{KB}{GB}$ and $\frac{KC}{GC}$.

We take

$$\frac{KK_b}{b} = \frac{abc}{2R(a^2+b^2+c^2)}$$

to be a known property of the Lemoine point (see, for example, Roger A. Johnson, *Advanced Euclidean Geometry*, Dover reprint (1960), page 214, paragraph 342) and obtain

$$a\frac{KA}{GA} = b\frac{KB}{GB} = c\frac{KC}{GC} = \frac{3abc}{a^2+b^2+c^2}. \tag{1}$$

On the other hand,

$$R^2 - OK^2 = (R - OK)(R + OK) = KA \cdot KA'' = KA(AA'' - KA). \tag{2}$$

Because $\Delta ABA' \sim \Delta AA''C$,

$$AA'' = \frac{bc}{A'A}. \tag{3}$$

From (1) we have

$$KA = \frac{bc \cdot 3GA}{a^2+b^2+c^2} = \frac{bc \cdot 2A'A}{a^2+b^2+c^2}. \tag{4}$$

Next, Stewart's theorem says that

$$A'A^2 = \frac{1}{4}\left(-a^2+2b^2+2c^2\right). \tag{5}$$

Putting together equations (2) through (5), we deduce that

$$\begin{aligned} R^2 - OK^2 &= \frac{2b^2c^2}{a^2+b^2+c^2} - \frac{4b^2c^2AA'^2}{(a^2+b^2+c^2)^2} \\ &= \frac{2b^2c^2}{a^2+b^2+c^2} - \frac{4b^2c^2}{(a^2+b^2+c^2)^2} \cdot \frac{1}{4}(-a^2+2b^2+2c^2) \\ &= \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}, \end{aligned}$$

as desired.

Also solved by TITU ZVONARU, Comăneşti, Romania; and the proposer.