SKOLIAD No. 140

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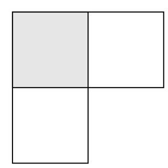
Skoliad has joined **Mathematical Mayhem** *which is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time Skoliad will be discontinued in* **Crux**. *New Skoliad problems, and their solutions, will appear in* **Mathematical Mayhem** *when it is relaunched in 2013.*

In this issue we present the solutions to the National Bank of New Zealand Junior Mathematics Competition, 2010. given in Skoliad 131 at [2011:65–71].

1. Rebecca is holding a seminar at the place at which she works. She wants to create an unbroken ring of tables, using a set of identical tables shaped like regular polygons. (In a regular polygon, all sides have the same length, and all angles are equal. Squares and equilateral triangles are regular.) Each table must have two sides which completely coincide with the sides of other tables, such as the shaded square table seen to the right. Rebecca plans to put items on display inside the ring where everyone can see them.



(If you cannot name a shape in this question, just give the number of sides. For example, if you think the shape has 235 sides, but don't know the name, just call it a 235-gon—that isn't an answer to any of the parts.)

1. Rebecca first decides to use identical square tables. What is the minimum number of square tables placed beside each other so that there is an empty space in the middle?

2. If Rebecca uses the minimal number of square tables, what shape is left bare in the middle?

3. Rebecca considers using octagon (eight sides) shaped tables.

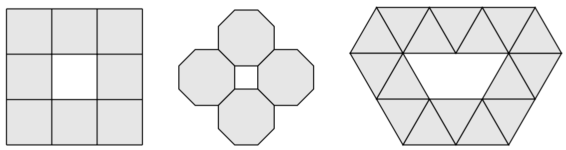
   (a) What is the minimum number of octagonal tables which Rebecca must have in order for there to be a bare space in the middle so that the tables form an enclosure?

   (b) What is the name given to the bare shape in the middle? If you can't name it, giving the number of sides will be sufficient.

4. Apart from squares and octagons, are there any other shaped tables possible? If there are any, name one. If there isn't, say so.

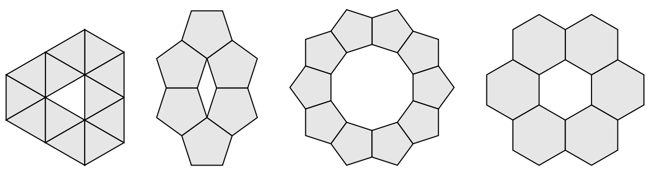
Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Rebecca needs at least eight square tables arranged as in the left-hand figure below. This leaves a square in the middle. If Rebecca uses regular octagons, she needs four arranged as in the middle figure below. This also leaves a square in the middle. Rebecca could also use equilateral triangles arranged as in the right-hand figure below.

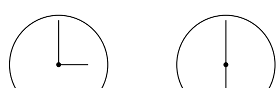


Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; JANICE LEW, student, École Alpha Secondary School, Burnaby, BC; KATIE PINTER, student, École Capitol Hill Elementary School, Burnaby, BC; SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC; and NELSON TAM, student, John Knox Christian School, Burnaby, BC.

Our solver uses sixteen equilateral triangles, but Rebecca could make do with twelve arranged as in the left-hand figure below. (Minimality was, of course, not part of the challenge.) Other solvers found more possible arrangements:



2. An analogue clock displays the time with the use of two hands. Every hour the minute hand rotates 360 degrees, while the hour hand (which is shorter than the minute hand) rotates 360 degrees over a 12-hour period. Two example times are shown below:

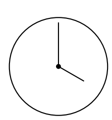


3 o'clock   6 o'clock

1. Draw a clock face which shows 9 o'clock. Make sure the hour hand is shorter than the minute hand.

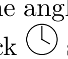
2. What is the angle between the two hands at both 3 o'clock and 9 o'clock?

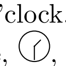
3. What time to the closest hour (and minute) does the following clock face show?



4. What is the angle between the two hands at the following times?
   (a) 1 o'clock.
   (b) 2 o'clock.
   (c) Half past one.
5. At what time (to the nearest minute) between 7 and 8 o'clock do the hands meet?

Solution by Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.

At 9 o'clock, an analogue clock looks like , and the angle between the two hands is 90°. At 3 o'clock, the angle is also 90°. The clock  shows 4:00.

At 1 o'clock, , the minute hand points straight up while the hour hand has moved $\frac{1}{12}$ of a turn. Therefore the angle between the hands is $\frac{1}{12} \cdot 360^\circ = 30^\circ$ at 1 o'clock. Likewise, at 2 o'clock the angle between the hands is 60°. At half past one, , the minute hand points straight down while the minute hand is exactly half way between 12 and 3, so half way between 0° and 90° from up, so 45° from up. Therefore the angle between the hands is $180^\circ - 45^\circ = 135^\circ$ at 1:30.

The minute hand makes one turn in 60 minutes, so it moves 360° in 60 minutes, so it moves 6° per minute. Likewise, the hour hand makes one turn in 12 hours, so it moves 360° in 720 minutes, so it moves $\frac{1}{2}^\circ$ per minute. If the time is x minutes past 7 o'clock, then the minute hand has moved $6x$ degrees from up, and the hour hand has moved $(7 \cdot 60 + x)\frac{1}{2}$ degrees from up. Therefore $6x = (7 \cdot 60 + x)\frac{1}{2}$ when the two hands meet, so $12x = 420 + x$, so $11x = 420$, so $x = \frac{420}{11} \approx 38.2$. Thus the two hands meet at approximately 7:38.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; JANICE LEW, student, École Alpha Secondary School, Burnaby, BC; KATIE PINTER, student, École Capitol Hill Elementary School, Burnaby, BC; and NELSON TAM, student, John Knox Christian School, Burnaby, BC.

3. A six-digit number "*abcdef*" is formed using each of the digits 1, 2, 3, 4, 5, and 6 once and only once so that "*abcdef*" is a multiple of 6, "*abcde*" is a multiple of 5, "*abcd*" is a multiple of 4, "*abc*" is a multiple of 3, and "*ab*" is a multiple of 2.

1. Find a solution for "*abcdef*." Show key working.
2. Is the solution you found unique (the only possible one)? If it is, briefly explain why. If it isn't, give another solution.

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Since "*abcde*" is divisible by 5, $e = 5$. Since "*ab*", "*abcd*", and "*abcdef*" are all divisible by even numbers, "*ab*", "*abcd*", and "*abcdef*" must themselves be even, so b, d, and f are even. Thus $\{b, d, f\} = \{2, 4, 6\}$ and $\{a, c\} = \{1, 3\}$.

If $a = 1$ and $c = 3$, the digit sum of "abc" is $4 + b$. Since "abc" is divisible by 3, this digit sum must be divisible by 3, so $b = 2$. Since "$abcd$" is divisible by 4, "cd" is divisible by 4. Since $c = 3$ and d is either 4 or 6, $d = 6$. Then f must be 4, and "$abcdef$" equals 123654.

If $a = 3$ and $c = 1$, the digit sum of "abc" is still $4 + b$, so again $b = 2$. The argument in the previous paragraph now leads to the solution 321654. Thus the problem admits two solutions.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

4. A 3×2 rectangle is divided up into six equal squares, each containing a bug. When a bell rings, the bugs jump either horizontally or vertically (they cannot jump diagonally and they stay within the rectangle) into a square adjacent to their previous square in any direction, although you cannot know in advance which exact square they will jump into. Every bug changes square; no bug stays put.

As an example, the ordered sextuplet $(1, 1, 1, 1, 1, 1)$ (where this represents the result, not the movement) represents the situation where every bug jumped so that each square still had one bug in it (it could happen). Alternatively, two bugs could also land in the same square. An example (not the only way this could happen) of this might be represented by $(2, 2, 1, 0, 0, 1)$ —see the diagram to the right. The first number in the sextuplet represents a corner square, the second represents a square on the middle of a side, and so on.

2	**2**	**1**
0	**0**	**1**

1. What is the average number of bugs per square in the 3 by 2 rectangle no matter how the bugs jump?

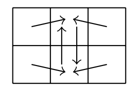
2. From the initial situation of one bug in every square, is it possible for three bugs to end up in the same square if the bell rings only once? If you think it is, write an ordered sextuplet like the two above where this could happen. If you think it can't happen, briefly explain why not.

3. From the initial situation of one bug in every square, it is certainly not possible in a 3×2 rectangle for four bugs to end up in the same square if the bell rings only once. Write down the dimensions of the smallest rectangle for which it would be possible.

4. From the initial situation of one bug in every square, five bugs can never end up in the same square if the bell rings only once, no matter the size of the rectangle. In a few words, explain why not.

5. In the 3×2 case, how many non-unique sextuplets (like $(1, 1, 1, 1, 1, 1)$) are possible from the initial situation of one bug in every square, if the bell rings only once? You do not have to list them, although you might like to.

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.



With six bugs and six squares, the average number of bugs per square is one. If the six bugs jump as in the figure to the right, there will be three bugs in each of the two middle squares after the first bell. To have four bugs in a square after the first bell, that square must have four neighbours (and all the neighbouring bugs must jump in, since the bug originally in any square must jump out). The smallest rectangle is therefore 3×3.

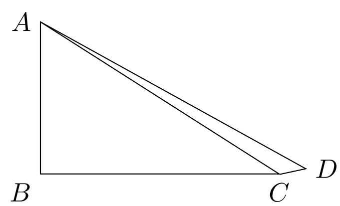
Each square has at most four neighbours, so at most four bugs can jump in at the first bell. The bug originally in any square must jump out at the first bell. Thus no square can contain five (or more) bugs after the first bell.

2	3	2
2	3	2

The bugs in the corner squares have two choices each for where to jump to. The bugs in the middle squares have three choices each. The figure lists the number of choices. The total number of jump-patterns is therefore $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^4 \cdot 3^2 = 144$.

Also solved by ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

5. Pania and Rangi exercise weekly by running around two paddocks on their father's farm near Kakanui from A to B to C to D and then back to A (see the diagram). In a direct line from A to C, the distance is 6250 m. AB is shorter than BC.



1. If $\triangle ABC$ is a right angled triangle in the ratio of $3:4:5$, with B at the right angle, find the lengths of the sides.

2. If $\triangle ABC$ is a right angled triangle in the ratio of $3:4:5$, with B at the right angle, find the size of $\angle CAB$ to one decimal place.

3. The angle at B is in fact a right angle, and AB and BC are whole metres in length, but the sides are not in the ratio of $3:4:5$. Find possible lengths for AB and BC.

4. The angle at D is not a right angle but is 40°, and CD is 600 m. Use this information to find the length of AD.

Hint: In any triangle XYZ, the following rules apply:

$$\text{Sine Law:} \quad \frac{x}{\sin X} = \frac{y}{\sin Y} = \frac{z}{\sin Z},$$
$$\text{Cosine Law:} \quad x^2 = y^2 + z^2 - 2yz \cos X,$$

where side x is opposite to angle X, side y is opposite to angle Y, and side z is opposite to angle Z.

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

If $|AB| : |BC| : |AC| = 3 : 4 : 5$, then $\frac{|AB|}{3} = \frac{|BC|}{4} = \frac{|AC|}{5} = \frac{6250}{5} = 1250$. Therefore $|AB| = 3 \cdot 1250 = 3750$ and $|BC| = 4 \cdot 1250 = 5000$. Moreover, $\tan \angle CAB = \frac{|BC|}{|AB|} = \frac{5000}{3750} = \frac{4}{3}$, so $\angle CAB = \tan^{-1}(\frac{4}{3}) \approx 53.1°$.

If $\angle ABC = 90°$, then $|AB|^2 + |BC|^2 = |AC|^2$ by the Pythagorean Theorem. The task therefore is to find a Pythagorean triple where the largest part is a divisor of $|AC| = 6250 = 2 \cdot 5^5$ other than 5. To that end, note that $(n^2 + m^2)^2 = n^4 + 2n^2m^2 + m^4$ and that $(n^2 - m^2)^2 = n^4 - 2n^2m^2 + m^4$. Thus $(n^2 + m^2)^2 - (n^2 - m^2)^2 = 4n^2m^2 = (2nm)^2$, so

$$(n^2 + m^2)^2 = (2nm)^2 + (n^2 - m^2)^2 \,.$$

Substituting integers for n and m in this equation will clearly generate many Pythagorean triples.

n	m	$n^2 + m^2$	$2nm$	$n^2 - m^2$
2	1	5	4	3
3	1	10	6	8
3	2	13	12	5
4	1	17	8	15
4	2	20	16	12
4	3	25	24	7
5	1	26	10	24
5	2	29	20	21
⋮	⋮	⋮	⋮	⋮

The triples $10 : 6 : 8$ and $20 : 16 : 12$ both reduce to the $3 : 4 : 5$ triple (in some order), but $25 : 24 : 7$ works out: $|AC| = 25 \cdot 250 = 6250$, $|AB| = 7 \cdot 250 = 1750$, and $|BC| = 24 \cdot 250 = 6000$.

Cosine Law in $\triangle ACD$ yields that

$$|AC|^2 = |AD|^2 + |CD|^2 - 2|AD||CD| \cos \angle ADC \,,$$

so if $\angle ADC = 40°$ and $|CD| = 600$, then $6250^2 = |AD|^2 + 600^2 - 2|AD|600 \cos 40°$, so $0 = |AD|^2 - 919.25333|AD| - 38702500$. Solving this with the Quadratic Formula yields that $|AD| = -5778.46$ (impossible) or $|AD| = 6697.72$.

Our solver's formula for generating Pythagorean triples is called Euclid's Formula. *This very useful formula generates all the interesting Pythagorean triples. The Wikipedia explains what we here mean by* interesting.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.