PROBLEM OF THE MONTH

No. 1

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*This column is dedicated to the memory of former **CRUX with MAYHEM** Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students; other people's students, through his work on mathematics contests and outreach programs; and the readers of **CRUX with MAYHEM**. The problem of the month is a short article that features a problem and its solution that we know Jim would have liked.*

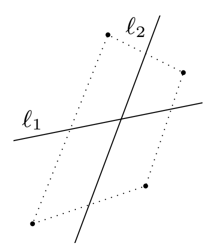
As an undergraduate student, I was fortunate to have Ross Honsberger as a professor a couple of times. One of the classes was on problem solving, and, in the first class, Professor Honsberger presented us with 100 problems. The remainder of the semester was spent solving and discussing the problems and the techniques used to solve them. One that remains in my memory is the following.

> *One million points are drawn in a plane such that no three are collinear. Prove that you can draw a line such that half the points are on one side of the line and half are on the other.*

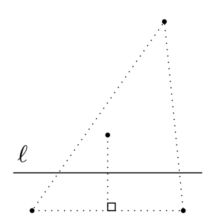
I remember being perplexed by this when I first saw it. I probably started looking at easier cases. The easiest case, with two points, is easy, we can choose the perpendicular bisector of the segment joining the two points.

Let's consider the next case with 4 points. We have two cases:

Case 1: The four points are vertices of a convex quadrilateral. In this case, note that if we draw the line through the midpoints of opposite sides, we get the desired result (as a matter of fact, a line through any two points on opposite sides works, as long as you don't use one of the original points). In the diagram ℓ_1 and ℓ_2 are lines that satisfy the condition.



Case 2: The four points are not the vertices of a convex quadrilateral, thus one of the points would be inside the triangle with vertices at the other three points. In this case, if we drop a perpendicular from the point on the inside to one of the sides, and then create the perpendicular bisector of the segment from the inside point to the base of the perpendicular, it has the desired property. In the diagram below ℓ is one of three such lines.



It doesn't take long to see that things will get very complex very quickly. In the case of of six points, we will get 4 cases corresponding to when the *convex hull* (the smallest convex shape that contains all the points) being a hexagon, a pentagon, a quadrilateral or a triangle. As the number of points grows, so will the number of cases. We need a new strategy.

Since we have showed that it works for 2 and 4 points, we may try an inductive proof. Assuming that we have shown that the process is possible for $2n$ points, for some value of n, now we will look at the case of $2n + 2$ points. By the induction hypothesis, each collection of $2n$ points has a line that satisfies the desired property. If, for some collection of $2n$ points, with line ℓ that satisfies the property, the extra 2 points are on opposite sides of ℓ, we are done. Unfortunately, there is no guarantee that this can be done in general.

We may be able to complete the induction proof with an "elementary" argument, but that could be quite long and complex. The solution to the problem, which I didn't produce at the time, still brings tears to my eyes because of its beauty.

Solution: Construct a line so that all points are on one side and the line is skew to all possible lines through every possible pair of points. Then, if we translate this line towards the points, it will encounter them one at a time. Thus we can move the line until it has passed through 500 000 points. After the line has passed through 500 000 points, but before it hits the 500 001$^{\text{st}}$ point, it satisfies the condition of the problem.

So there it is, a seemingly "obvious" property shown to be true by looking at it from the right perspective. Thank you Professor Honsberger!

You may want to try your hand at a similar problem given to me by Dr. Robert Craigen from the University of Manitoba.

> *One hundred planets in a solar system are moving in some complex pattern that keeps them from crashing into each other. Each planet has radius exactly 1000 km. Planets are lit by the following process: whenever a point on the surface of one planet can be seen from from a point on the surface of another, both points are lit. Conversely it is dark at any point on any planet from which none of the other planets are visible. Prove that the total dark area on all the planets together is a constant, and determine that constant.*