OC83. Sur un demi-cercle de diamètre $|AB| = d$, on donne deux points C et D tels que $|BC| = |CD| = a$ et $|DA| = b$, où a, b, d sont trois entiers positifs distincts. Trouver le minimum possible de la valeur de d.

OC84. Soit m et n deux entiers positifs. Montrer qu'il existe une infinité de couples d'entiers positifs relativement premiers (a, b) tels que

$$a + b \mid am^a + bn^b.$$

OC85. Montrer que pour tout entier positif d, il existe une infinité d'entiers positifs n tels que $d(n!) - 1$ est un nombre composé.

OLYMPIAD SOLUTIONS

OC21. A sequence of real numbers $\{a_n\}$ is defined by $a_0 \neq 0, 1$, $a_1 = 1 - a_0$, and $a_{n+1} = 1 - a_n(1 - a_n)$ for $n = 1, 2, \cdots$. Prove that for any positive integer n, we have

$$a_0 a_1 \cdots a_n \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) = 1\,.$$

(*Originally question #1 from the 2008 China Western Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France ; Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Oliver Geupel, Brühl, NRW, Germany ; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina ; Henry Ricardo, Tappan, NY, USA ; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Cománeşti, Romania. We give the solution of Krimker.

Since the equation $x^2 - x + 1 = 0$ doesn't have any real solutions, it is clear that for all n we have $a_n \neq 0$. It is also easy to prove by induction that $a_n \neq 1$.

Now, using

$$a_n = \frac{1 - a_{n+1}}{1 - a_n}\,,$$

we get

$$a_0 a_1 \cdots a_n = (1 - a_1) \frac{1 - a_2}{1 - a_1} \frac{1 - a_3}{1 - a_2} \cdots \frac{1 - a_{n+1}}{1 - a_n} = 1 - a_{n+1}$$

We are now ready to prove the statement by induction. Since

$$a_0 a_1 \left(\frac{1}{a_0} + \frac{1}{a_1} \right) = a_0 + a_1 = 1,$$

the statement is true for $n = 1$. Next assume that the statement is true for some value of n, then

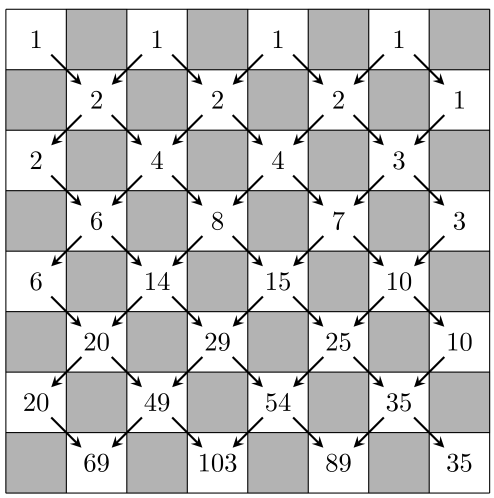
$$\begin{aligned} a_0 a_1 \cdots a_n a_{n+1} & \left(\frac{1}{a_0} + \frac{1}{a_1} \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}} \right) \\ &= a_0 a_1 \cdots a_n a_{n+1} \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) + a_0 a_1 \cdots a_n \\ &= a_{n+1} + 1 - a_{n+1} = 1 \end{aligned}$$

which completes the proof.

OC22. Consider a standard 8×8 chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?
(*Originally question #1 from the 2008/9 British Mathematical Olympiad, Round 1.*)

Solved by Geneviève Lalonde, Massey, ON.

We can label each white square with the number of (distinct) paths that reach the square from the top. As a result, the number of paths to each white square is the sum of the number of paths to the two white squares above it, or equal to the number of paths to the only white square above it.



Thus there are a total of 296 zig-zag paths.

OC23. Determine all nonnegative integers n such that

$$n(n-20)(n-40)(n-60)\cdots r + 2009$$

is a perfect square where r is the remainder when n is divided by 20.
(*Originally question #2 from the 40th Austrian Mathematical Olympiad, National Competition, Final Round (G. Baron, Vienna).*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA and Titu Zvonaru, Cománeşti, Romania. We give the solution of Curtis, modified by the editor.

We claim that $n = 16$ is the only solution.
First let's observe that if $n \geq 40$ then

$$n(n-20)(n-40) \equiv n(n+1)(n+2) \equiv 0 \pmod 3 .$$

Thus

$$n(n-20)(n-40)\cdots r+2009 \equiv 2 \pmod 3 .$$

Hence, no $n \geq 40$ produces a perfect square.

If $20 \leq n < 39$ then

$$n(n-20)+2009 = k^2 \Leftrightarrow (n-10)^2 = k^2 - 1909 .$$

But then

$$10^2 \leq k^2 - 1909 \leq 19^2 \Rightarrow 2009 \leq k^2 \leq 2270 \Rightarrow 44.8 \leq k \leq 47.6 .$$

If $k = 45$ then $(n-1)^2 = 116$ which is not possible.
If $k = 46$ then $(n-1)^2 = 207$ which is not possible.
If $k = 47$ then $(n-1)^2 = 300$ which is not possible.
Last, if $n < 20$ then we have

$$n + 2009 = k^2 .$$

Thus

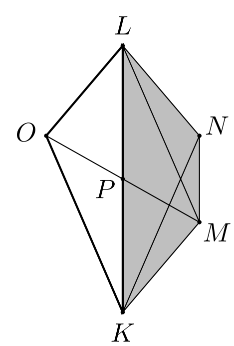
$$2009 \leq k^2 < 2029 \Rightarrow k = 45 \Rightarrow n = 16.$$

OC24. Let O be the circumcentre of the triangle ABC. Let K and L be the intersection points of the circumcircles of the triangles BOC and AOC with the bisectors of the angles at A and B respectively. Let P be the midpoint of KL, M symmetrical to O relative to P and N symmetrical to O relative to KL. Prove that $KLMN$ is cyclic.
(*Originally question #2 from the 16th Macedonian Mathematical Olympiad.*)

Similar solutions by Michel Bataille, Rouen, France; Mihaï-Ioan Stoënescu, Bischwiller, France and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

We will solve the more general problem:

Let KOL be any triangle and let P be the midpoint of KL. Let M be the symmetrical image of O relative to P and N the symmetrical image of O relative to KL. Then $KLMN$ is cyclic.



If $LO = OK$, then $M = N$ and $KLMN$ is a triangle.

Otherwise, the diagonals of $LMKO$ halve each other, and hence $LMKO$ is a parallelogram. Thus

$$\angle LMK = \angle LOK. \qquad (1)$$

By symmetry we also have

$$\angle LNK = \angle LOK. \qquad (2)$$

Combining (1) and (2) we get $\angle LMK = \angle LNK$ which proves the desired result.

OC25. Show that the inequality $3^{n^2} > (n!)^4$ holds for all positive integers n.
(*Originally question #1 from the 40th Austrian Mathematical Olympiad, National Competition, Final Round (G. Baron, Vienna).*)

We provide the similar solutions by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Henry Ricardo, Tappan, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA and Titu Zvonaru, Cománeşti, Romania.

We first prove by induction the proposition $P_1(n)$ that $3^{2n+1} > (n+1)^4$ for all $n \geq 1$.

As $3^3 > 2^4$, $P_1(1)$ is true. Suppose $P_1(n)$ is true for some $n \geq 1$. Since $\left(\frac{n+2}{n+1}\right)^2 = \left(1 + \frac{1}{n+1}\right)^2 \leq \left(1 + \frac{1}{n+1}\right)^{n+1} < e < 3$, we have $9 > \frac{(n+2)^4}{(n+1)^4}$, so

$$\begin{aligned} 3^{2n+3} &= 3^2 3^{2n+1} > 9(n+1)^4 \\ &> \frac{(n+2)^4}{(n+1)^4}(n+1)^4 = (n+2)^4, \end{aligned}$$

hence $P_1(n+1)$ is true and thus $P_1(n)$ is true for all $n \geq 1$.

Now we can prove the original problem by induction. Let $P_2(n)$ be the proposition that $3^{n^2} > (n!)^4$, then $P_2(1)$ is clearly true since: $3^1 > 1^4$. Suppose $P_2(n)$ is true for some $n \geq 1$, then

$$\begin{aligned} 3^{(n+1)^2} &= 3^{n^2} 3^{2n+1} \\ &> (n!)^4 (n+1)^4 = [(n+1)!]^4. \end{aligned}$$

Thus $P_2(n+1)$ is true. This solves the problem.

Editor's Note: The inequality $3^{2n+1} > (n+1)^4$ has a very simple combinatorial proof. Let $A = \{1,2,3,\ldots,n+1\} \times \{1,2,3,\ldots,n+1\} \times \{1,2,3,\ldots,n+1\} \times \{1,2,3,\ldots,n+1\}$ and $B = \{f : \{1,2,3,\ldots,2n+1\} \to \{0,1,2\}\}$. Then it is easy to construct an injective function from A to B.

For example: $(a,b,c,d) \to f$ where $f : \{1,2,3,\ldots,2n+1\} \to \{0,1,2\}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 2 & \text{if } x = b \\ 1 & \text{if } x = n+1+c \\ 2 & \text{if } x = n+1+d \\ 0 & \text{otherwise} \end{cases}$$

is such a function. [Note that $c = n+1$ if and only if f only takes the value 1 once, and $d = n+1$ if and only if f only takes the value 2 once.]