

THE OLYMPIAD CORNER

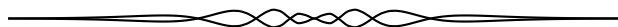
No. 303

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The solutions to the problems are due to the editor by 1 November 2013.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.



OC81. Find all triplets (x, y, z) of integers that satisfy

$$x^4 + x^2 = 7^z y^2.$$

OC82. The area and the perimeter of the triangle with sides 6, 8, 10 are equal. Find all triangles with integral sides whose area and perimeter are equal.

OC83. On a semicircle with diameter $|AB| = d$ we are given points C and D such that $|BC| = |CD| = a$ and $|DA| = b$, where a, b, d are different positive integers. Find the minimum possible value of d .

OC84. Let m, n be positive integers. Prove that there exist infinitely many pairs of relatively prime positive integers (a, b) such that

$$a + b \mid am^a + bn^b.$$

OC85. For any positive integer d , prove there are infinitely many positive integers n such that $d(n!) - 1$ is a composite number.



OC81. Trouver tous les triplets d'entiers (x, y, z) satisfaisant

$$x^4 + x^2 = 7^z y^2.$$

OC82. Dans le triangle dont les côtés mesurent 6, 8, 10, l'aire et le périmètre sont égaux. Trouver tous les triangles dont les côtés sont entiers et dont l'aire et le périmètre sont égaux.

OC83. Sur un demi-cercle de diamètre $|AB| = d$, on donne deux points C et D tels que $|BC| = |CD| = a$ et $|DA| = b$, où a, b, d sont trois entiers positifs distincts. Trouver le minimum possible de la valeur de d .

OC84. Soit m et n deux entiers positifs. Montrer qu'il existe une infinité de couples d'entiers positifs relativement premiers (a, b) tels que

$$a + b \mid am^a + bn^b.$$

OC85. Montrer que pour tout entier positif d , il existe une infinité d'entiers positifs n tels que $d(n!) - 1$ est un nombre composé.

OLYMPIAD SOLUTIONS

OC21. A sequence of real numbers $\{a_n\}$ is defined by $a_0 \neq 0, 1$, $a_1 = 1 - a_0$, and $a_{n+1} = 1 - a_n(1 - a_n)$ for $n = 1, 2, \dots$. Prove that for any positive integer n , we have

$$a_0 a_1 \cdots a_n \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) = 1.$$

(Originally question #1 from the 2008 China Western Mathematical Olympiad.)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Henry Ricardo, Tappan, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Comănești, Romania. We give the solution of Krimker.

Since the equation $x^2 - x + 1 = 0$ doesn't have any real solutions, it is clear that for all n we have $a_n \neq 0$. It is also easy to prove by induction that $a_n \neq 1$.

Now, using

$$a_n = \frac{1 - a_{n+1}}{1 - a_n},$$

we get

$$a_0 a_1 \cdots a_n = (1 - a_1) \frac{1 - a_2}{1 - a_1} \frac{1 - a_3}{1 - a_2} \cdots \frac{1 - a_{n+1}}{1 - a_n} = 1 - a_{n+1}$$

We are now ready to prove the statement by induction. Since

$$a_0 a_1 \left(\frac{1}{a_0} + \frac{1}{a_1} \right) = a_0 + a_1 = 1,$$

the statement is true for $n = 1$. Next assume that the statement is true for some value of n , then

$$\begin{aligned} a_0 a_1 \cdots a_n a_{n+1} & \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_{n+1}} \right) \\ & = a_0 a_1 \cdots a_n a_{n+1} \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) + a_0 a_1 \cdots a_n \\ & = a_{n+1} + 1 - a_{n+1} = 1 \end{aligned}$$