FOCUS ON . . .

No. 2

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The Geometry Behind the Scene

Introduction

The algebraic presentation of some problems (and their solutions) can mask the connection with a geometric problem. For familiar examples, think of a diophantine equation, such as $x^2 - 4xy + 6y^2 - 2x - 20y = 29$ from Problem 2850 [2003 : 242 ; 2004 : 302], which can be simplified by introducing the centre of the corresponding ellipse; or recall, in the proof of an inequality, the link between a constraint such as $abc = a + b + c$ and the angles of a triangle. We will consider in detail two less obvious examples and unveil their geometric background.

First example: A hidden hyperboloid

Virgil Nicula's Problem 3309 [2008 : 45,48 ; 2009 : 59], asks for a necessary and sufficient condition on nonzero real numbers α, β, γ for the system

$$\alpha x + \beta y + \gamma z = 1$$
$$xy + yz + zx = 1$$

to have a unique solution. The featured solution (by G. Tsapakidis) is nice and elementary, resting on the properties of quadratic equations. However, faced with this problem, I, and probably many other solvers, cannot help seeing a plane $\mathcal{P}$ in $\alpha x + \beta y + \gamma z = 1$ and a quadric $\mathcal{H}$ in $xy + yz + zx = 1$ (namely a hyperboloid with two sheets). I suspect that this is also the origin of the problem! Once this observation has been made, a less elementary but more illuminating solution can follow: the uniqueness of a solution just means that $\mathcal{P}$ is tangent to the hyperboloid at some point (x_0, y_0, z_0). Since the equation of the plane tangent to $\mathcal{H}$ at (x_0, y_0, z_0) is

$$(x - x_0)(y_0 + z_0) + (y - y_0)(z_0 + x_0) + (z - z_0)(x_0 + y_0) = 0,$$

$\mathcal{P}$ is tangent to $\mathcal{H}$ if and only if

$$\frac{y_0 + z_0}{\alpha} = \frac{z_0 + x_0}{\beta} = \frac{x_0 + y_0}{\gamma} \tag{1}$$

for some (x_0, y_0, z_0) such that

$$x_0y_0 + y_0z_0 + z_0x_0 = 1 \tag{2}$$

and

$$\alpha x_0 + \beta y_0 + \gamma z_0 = 1. \tag{3}$$

Now, the common value of the ratios in (1) is

$$\frac{x_0(y_0+z_0)+y_0(z_0+x_0)+z_0(x_0+y_0)}{\alpha x_0+\beta y_0+\gamma z_0}=2,$$

and solving (1) for x_0, y_0, z_0, we obtain

$$x_0=\gamma+\beta-\alpha, \quad y_0=\alpha+\gamma-\beta, \quad z_0=\alpha+\beta-\gamma. \tag{4}$$

Plugging these values in either of the conditions (2) or (3) yields the desired relation

$$\alpha^2+\beta^2+\gamma^2+1=2(\alpha\beta+\beta\gamma+\gamma\alpha).$$

A sphere in a cylinder

Our second example is Problem 3455 [2009 : 326, 328 ; 2010 : 344], a problem that I made up (here I am quite sure of its geometric origin!). Kee-Wai Lau's featured solution is short and elegant but does not convey the geometrical flavor. We offer the following generalization and a solution based on the geometrical ideas that underlie the problem:

Given real numbers a, b, c, not all zero, and $k > 0$, find the minimal value of $x^2+y^2+z^2$ over all real numbers x, y, z subject to $f(x,y,z) \geq k^2$ where

$$f(x,y,z)=(b^2+c^2)x^2+(c^2+a^2)y^2+(a^2+b^2)z^2-2abxy-2bcyz-2cazx.$$

Noticing that

$$f(x,y,z)=(bx-ay)^2+(cy-bz)^2+(az-cx)^2=\|\overrightarrow{\Omega}\times\overrightarrow{OM}\|^2 \tag{5}$$

where $\overrightarrow{\Omega}(a,b,c)$ and $M(x,y,z)$ in a system of axes with origin O in space, we see that $f(x,y,z)=k^2$ is the equation of a cylinder $\mathcal{C}$ whose generators are parallel to $\overrightarrow{\Omega}$. Thus, the problem can be interpreted as the search for the minimal distance between O and a point M exterior to or on the cylinder $\mathcal{C}$ (alternatively for the biggest sphere with centre O entirely contained in $\mathcal{C}$). Clearly, this minimal distance OM is attained when M is on the normal section through O, that is, the circle intersection of $\mathcal{C}$ with the plane $ax+by+cz=0$.

With the help of (5), this approach can be specified as follows:
Since $\|\overrightarrow{\Omega}\times\overrightarrow{OM}\|^2=\|\overrightarrow{\Omega}\|^2\|\overrightarrow{OM}\|^2\sin^2\theta$ where θ is the angle of $\overrightarrow{\Omega}$ and $\overrightarrow{OM}$, we have

$$x^2+y^2+z^2=\|\overrightarrow{OM}\|^2 \geq \frac{\|\overrightarrow{\Omega}\times\overrightarrow{OM}\|^2}{\|\overrightarrow{\Omega}\|^2}=\frac{f(x,y,z)}{a^2+b^2+c^2}$$

for all x, y, z and so

$$x^2+y^2+z^2 \geq \frac{k^2}{a^2+b^2+c^2}$$

when $f(x,y,z) \geq k^2$ with equality if M is on $\mathcal{C}$ and $\overrightarrow{OM}$ is orthogonal to $\overrightarrow{\Omega}$. Thus, the desired minimum is $\dfrac{k^2}{a^2+b^2+c^2}$ (and the biggest sphere with centre O interior to $\mathcal{C}$ is the one with radius $\dfrac{k}{\sqrt{a^2+b^2+c^2}}$).

An exercise

Perhaps the reader would like to interpret the following problem (slightly adapted from problem 11301 in[1]) in plane geometry and discover a variant of solution:

Show that for any complex numbers a, b, c,

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \leq \frac{9}{16}(|a|^2+|b|^2+|c|^2)^2.$$

Hint: note that

$$ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2) = (b-a)(a-c)(c-b)(a+b+c)$$

and considering a, b, c as the complex affixes of A, B, C, express $OA^2+OB^2+OC^2$ with the help of the isobarycentre of A, B, C.

References

[1] *Amer. Math. Monthly*, Vol. 116, No 1, January 2009, p. 85-6.