About the Japanese theorem

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Dedicated to the memory of the great professor,
Laurenţiu Panaitopol

Abstract

The aim of this paper is to present three new proofs of the Japanese Theorem and several applications.

1 Introduction

A *cyclic quadrilateral* (or *inscribed quadrilateral*) is a convex quadrilateral whose vertices all lie on a single circle. Given a cyclic quadrilateral $ABCD$, denote by O the circumcenter, R the circumradius, and by a, b, c, d, e, and f the lengths of the segments AB, BC, CD, DA, AC and BD respectively. Recall Ptolemy's Theorems [4, pages 62 and 85] for a cyclic quadrilateral $ABCD$:

$$ef = ac + bd \tag{1}$$

and

$$\frac{e}{f} = \frac{ad + bc}{ab + cd}. \tag{2}$$

Another interesting relation for cyclic quadrilaterals is given by the Japanese Theorem ([4]). This relates the radii of the incircles of the triangles BCD, CDA, DAB and ABC, denoted by r_a, r_b, r_c, and r_d respectively, in the following way:

$$r_a + r_c = r_b + r_d. \tag{3}$$

In [8], W. Reyes gave a proof of the Japanese Theorem using a result due to the French geometer Victor Thébault. Reyes mentioned that a proof of this theorem can be found in [3, Example 3.5(1), p. 43, 125-126]. In [9, p. 155], P. Yiu found a simple proof of the Japanese Theorem. In [5], D. Mihalca, I. Chiţescu and M. Chiriţă demonstrated (3) using the identity $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$, which is true in any triangle ABC, where r is the inradius of ABC, and in [7], M. E. Panaitopol and L. Panaitopol show that

$$r_a + r_c = R(\cos x + \cos y + \cos z + \cos u - 2) = r_b + r_d,$$

where $m(\overset{\frown}{AB}) = 2x$, $m(\overset{\frown}{BC}) = 2y$, $m(\overset{\frown}{CD}) = 2z$ and $m(\overset{\frown}{AD}) = 2u$. In this paper, we will give three new proofs.

2 MAIN RESULTS

Lemma 1 *If $ABCD$ is a cyclic quadrilateral, then* $\dfrac{e}{f} = \dfrac{(a+b+e)(c+d+e)}{(b+c+f)(a+d+f)}$.

Proof. From (2), we deduce the equality $abe + cde = adf + bcf$. Adding the same terms in both parts of this equality, we have

$$abe+cde+e^2f+aef+def+bef+cef+ef^2 = adf+bcf+ef^2+aef+def+bef+cef+e^2f.$$

But, from equation (1), we have $e^2f = e(ac + bd) = ace + bde$ and $ef^2 = f(ac + bd) = acf + bdf$. Therefore, we obtain

$$\begin{aligned} &abe + cde + ace + bde + aef + def + bef + cef + ef^2 \\ &\qquad = adf + bcf + acf + bdf + aef + def + bef + cef + e^2f, \end{aligned}$$

which means that $e(b+c+f)(a+d+f) = f(a+b+e)(c+d+e)$, and the Lemma follows. □

In the following we give a property of a cyclic quadrilateral which we use in proving the Japanese Theorem.

Theorem 1 *In any cyclic quadrilateral there is the following relation:*

$$r_a \cdot r_c \cdot e = r_b \cdot r_d \cdot f \tag{4}$$

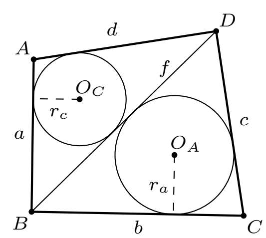

Figure 1

Proof. For triangles BCD and ABD, we write the equations [2, p. 11]

$$r_a = \frac{b+c-f}{2}\tan\frac{C}{2},\ r_c = \frac{a+d-f}{2}\tan\frac{A}{2}.$$

But $\tan\frac{A}{2} \cdot \tan\frac{C}{2} = 1$, because $A + C = \pi$. Therefore, we obtain

$$4r_ar_c = ab + cd + ac + bd - f(a+b+c+d) + f^2,$$

so from (1), we deduce

$$4r_ar_c = ab + cd + f(e + f - a - b - c - d).$$

Multiplying by e, we obtain

$$4r_ar_ce = e(ab + cd) + ef(e + f - a - b - c - d). \tag{5}$$

Similarly, we deduce that

$$4r_b r_d f = f(ad+bc) + ef(e+f-a-b-c-d). \tag{6}$$

Combining (2), (5) and (6) we obtain (4). □

G. Szöllősy, [6], proposed (7) below for a cyclic quadrilateral. We provide two new proofs of this relation.

Theorem 2 *In a cyclic quadrilateral, the identity*

$$\frac{abe}{a+b+e} + \frac{cde}{c+d+e} = \frac{bcf}{b+c+f} + \frac{adf}{a+d+f}, \tag{7}$$

holds.

Proof I. Let $m(\widehat{AB}) = 2x$, $m(\widehat{BC}) = 2y$, $m(\widehat{CD}) = 2z$ and $m(\widehat{AD}) = 2t$. Then $x+y+z+t=\pi$. By definition, $a = 2R\sin x$, $b = 2R\sin y$, $c = 2R\sin z$, $d = 2R\sin t$, $e = 2R\sin(x+y) = 2R\sin(z+t)$, $f = 2R\sin(x+t) = 2R\sin(y+z)$. Equation (7) now follows from the trigonometric identity

$$\begin{aligned}\frac{\sin\alpha\sin\beta\sin(\alpha+\beta)}{\sin\alpha+\sin\beta+\sin(\alpha+\beta)} &= 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\alpha+\beta}{2} \\ &= \left(\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right)\cos\frac{\alpha+\beta}{2},\end{aligned}$$

for any $\alpha, \beta \in \mathbb{R}$, with $\sin\alpha+\sin\beta+\sin(\alpha+\beta) \neq 0$. □

Proof II. From Lemma 1, we have

$$\frac{e}{(a+b+e)(c+d+e)} = \frac{f}{(b+c+f)(a+d+f)}. \tag{8}$$

From (2), $abe + cde = adf + bcf$, we obtain

$$ab(c+d+e) + cd(a+b+e) = bc(a+d+f) + ad(b+c+f). \tag{9}$$

Combining (8) and (9), we deduce

$$\frac{abe(c+d+e) + cde(a+b+e)}{(a+b+e)(c+d+e)} = \frac{bcf(a+d+f) + adf(b+c+f)}{(b+c+f)(a+d+f)}.$$

Consequently, we obtain (7). □

Next, we present three new proofs of the Japanese Theorem.

Theorem 3 *(The Japanese Theorem) Let $ABCD$ be a convex quadrilateral inscribed in a circle. Denote by r_a, r_b, r_c, and r_d the inradii of the triangles BCD, CDA, DAB, and ABC respectively. Then $r_a + r_c = r_b + r_d$.*

Proof I. Recall [4, Section 298i, p. 190] that for any triangle ABC with circumradius R and inradius r, we have the relation

$$r = \frac{abc}{2R(a+b+c)}.$$

In particular, for our four triangles we have

$$r_a = \frac{bcf}{2R(b+c+f)}, r_b = \frac{cde}{2R(c+d+e)}, r_c = \frac{adf}{2R(a+d+f)} \text{ and } r_d = \frac{abe}{2R(a+b+e)}.$$

The theorem then follows immediately from (7). □

Proof II. Applying the equations for the inradii that we used in the first proof to triangles BCD and ABD, we obtain

$$\begin{aligned} r_a + r_c &= r_a r_c \cdot \left(\frac{1}{r_a} + \frac{1}{r_c}\right) = \frac{r_a r_c}{f} \cdot \left(\frac{f}{r_a} + \frac{f}{r_c}\right) \\ &= \frac{r_a r_c}{f} \cdot \frac{2R}{abcd} \cdot [abc + abd + acd + bcd + f(ad + bc)]. \qquad (10) \end{aligned}$$

Similarly, for triangles CDA and ABC, we deduce

$$r_b + r_d = \frac{r_b r_d}{e} \cdot \frac{2R}{abcd} \cdot [abc + abd + acd + bcd + e(ab + cd)]. \qquad (11)$$

From Equation (2), $e(ab + cd) = f(ad + bc)$. Plug this together with Equation (4) into equations (10) and (11), and the theorem follows. □

Proof III. In the cyclic quadrilateral $ABCD$ we let I_a; I_b; I_c, and I_d denote the incenters of triangles BCD; DAC; ABD, and ABC respectively (see Figure 2).

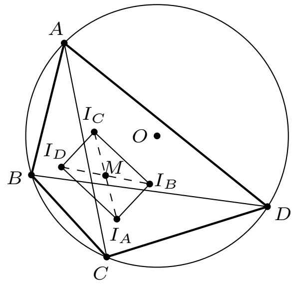

Figure 2

A theorem attributed to Fuhrmann [4, Section 422, p. 255] says that the quadrilateral $I_a I_b I_c I_d$ is a rectangle. See also [9, p. 154] for a neat proof. Let M be a point so that $I_a I_c \cap I_b I_d = \{M\}$, so M is the midpoint of the diagonals $I_a I_c$ and $I_b I_d$. The following theorem has been attributed to Apollonius [2, p. 6]: *In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects* *the third side*. We apply Apollonius's Theorem to the triangles I_aOI_c and I_bOI_d, where O is the circumcenter of the cyclic quadrilateral $ABCD$, and we obtain the relations $4OM^2 = 2(OI_a^2+OI_c^2)-I_aI_c^2$ and $4OM^2 = 2(OI_b^2+OI_d^2)-I_bI_d^2$, whence, and because $I_aI_c = I_bI_d$,

$$OI_a^2 + OI_c^2 = OI_b^2 + OI_d^2. \tag{12}$$

Euler's formula for the distance d between the circumcentre (O) and incentre (I) of a triangle is given by $d^2 = R^2-2Rr$, where R and r denote the circumradius and inradius respectively [2, p. 29]. For a proof using complex numbers we mention the book of T. Andreescu and D. Andrica [1]. In our case, the triangles ABC, BCD, CDA, DAB have the same circumcircle. In these triangles we apply Euler's relation. Hence, (12) becomes $R^2 - 2Rr_a + R^2 - 2Rr_c = R^2 - 2Rr_b + R^2 - 2Rr_d$, and the theorem follows. □

3 APPLICATIONS

If for a triangle ABC the points A', B', and C' are the points of contact between the sides BC, AC, and AB and the three excircles, respectively, then the segments AA', BB', and CC' meet at one point, which is called the *Nagel point*. Denote by O the circumcenter, I the incenter, N the Nagel point, R the circumradius, and r the inradius of ABC. An important distance is ON and it is given by

$$ON = R - 2r. \tag{13}$$

Equation (13) gives the geometric difference between the quantities involved in Euler's inequality $R \geq 2r$. A proof using complex numbers is given in the book of T. Andreescu and D. Andrica [1].

Application 1. *Let $ABCD$ be a convex quadrilateral inscribed in a circle with the center O. Denote by N_a, N_b, N_c, N_d the Nagel points of the triangles BCD, CDA, DAB, and ABC, respectively. Then the relation $ON_a + ON_c = ON_b + ON_d$ holds.*

Proof. From the Japanese Theorem, we have $r_a + r_c = r_b + r_d$. Therefore we obtain $R - 2r_a + R - 2r_c = R - 2r_b + R - 2r_d$. The statement of the Theorem now follows from (13). □

Our final application follows quickly from (3) and (4).

Application 2. *In any cyclic quadrilateral there are the following relations:*

$$f\left(\frac{1}{r_a} + \frac{1}{r_c}\right) = e\left(\frac{1}{r_b} + \frac{1}{r_d}\right)$$

and

$$e(r_a^2 + r_c^2) = f(r_b^2 + r_d^2).$$

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References

[1] T. Andreescu and D. Andrica, *Complex Numbers from A to .. Z*, Birkhauser, Boston-Basel-Berlin, 2006.

[2] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, The Mathematical Association of America, New Mathematical Library, 1967.

[3] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry*, Charles Babbage Research Centre, Manitoba, Canada, 1989.

[4] R. A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, New York, 1960.

[5] D. Mihalca,I. Chiţescu and M. Chiriţă, *Geometria patrulaterului*, Editura Teora, Bucureşti, 1998 (in Romanian).

[6] O. T. Pop, N. Minculete and M. Bencze, *An introduction to quadrilateral geometry* (preprint).

[7] M. E. Panaitopol and L. Panaitopol, *Probleme de geometrie rezolvate trigonometric*, Editura GIL, Zalău, 1994 (in Romanian).

[8] W. Reyes, An Application of Thébault's Theorem, *Forum Geometricorum*, **2** (2002) 183-185.

[9] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, Florida, 1998.
http://math.fau.edu/yiu/euclideangeometrynotes.pdf

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