# Mayhem Problems

*Please send your solutions to the problems in this edition by **1 August 2012**. Solutions received after this date will only be considered if there is time before publication of the solutions.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.*

*The editor thanks Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the problems from English into French.*

**M495**. *Proposed by the Mayhem Staff.*

All possible lines are drawn through the point $(0,0)$ and the points $(x,y)$, where $x$ and $y$ are whole numbers with $1 \le x, y \le 10$. How many distinct lines are drawn?

**M496**. *Proposed by Sally Li, student, Marc Garneau Collegiate Institute, Toronto, ON.*

Show that if we write the numbers from $1$ to $n$ around a circle, in any order, then, for all $x = 1, 2, \ldots, n$, we are guaranteed to find a block of $x$ consecutive numbers that add up to at least $\left\lceil \dfrac{x(n+1)}{2} \right\rceil$. Here $\lceil y \rceil$ is the ceiling function, that is, the least integer greater than or equal to $y$. So $\lceil 6.2 \rceil = 7$, $\lceil \pi \rceil = 4$, $\lceil -8.3 \rceil = -8$ and $\lceil 10 \rceil = 10$.

**M497**. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

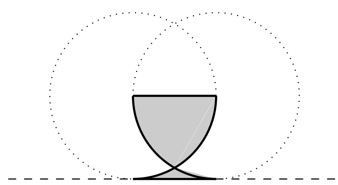
Find all integers $a$, $b$, $c$ where $c$ is a prime number such that $a^b + c$ and $a^b - c$ are both perfect squares.

**M498**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Right triangle $ABC$ has its right angle at $C$. The two sides $CB$ and $CA$ are of integer length. Determine the condition for the radius of the incircle of triangle $ABC$ to be a rational number.

**M499**. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Two circles of radius $1$ are drawn so that each circle passes through the centre of the other circle. Find the area of the goblet like region contained between the common radius, the circumferences and one of the common tangents as shown in the diagram to the right.



**M500**. *Proposed by Edward T.H. Wang and Dexter S.Y. Wei, Wilfrid Laurier University, Waterloo, ON.*

Let $\mathbb{N}$ denote the set of natural numbers.

(a) Show that if $n \in \mathbb{N}$, there do not exist $a, b \in \mathbb{N}$ such that $\dfrac{[a,b]}{a+b} = n$, where $[a,b]$ denotes the least common multiple of $a$ and $b$.

(b) Show that for any $n \in \mathbb{N}$, there exists infinitely many triples $(a,b,c)$ of natural numbers such that $\dfrac{[a,b,c]}{a+b+c} = n$, where $[a,b,c]$ denotes the least common multiple of $a$, $b$ and $c$.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**M495**. *Proposé par l'Équipe de Mayhem.*

Toutes les droites possibles sont tracées à partir du point $(0,0)$ et des points $(x,y)$, où $x$ et $y$ sont des entiers tels que $1 \le x, y \le 10$. Combien de droites distinctes ont été tracées ?

**M496**. *Proposé par Sally Li, Institut collégial Marc Garneau, Toronto ON.*

Démontrer que si on place tous les entiers de $1$ à $n$ autour d'un cercle dans un ordre quelconque alors, pour tout $x = 1, 2, \ldots, n$, on pourra certainement trouver un bloc de $x$ nombres consécutifs dont la somme sera d'au moins $\left\lceil \dfrac{x(n+1)}{2} \right\rceil$, $\lceil y \rceil$ désignant la partie "plafond" de $y$, c'est-à-dire le plus petit entier plus grand ou égal à $y$. Ainsi $\lceil 6.2 \rceil = 7$, $\lceil \pi \rceil = 4$, $\lceil -8.3 \rceil = -8$ and $\lceil 10 \rceil = 10$.

**M497**. *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

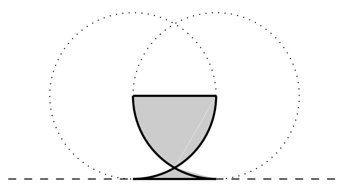
Déterminer tous les entiers $a$, $b$ et $c$ où $c$ est un nombre premier tel que $a^b + c$ et $a^b - c$ sont tous les deux des carrés d'entiers.

**M498**. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Le triangle rectangle $ABC$ a l'angle droit à $C$ ; les deux côtés $CB$ et $CA$ sont de longueurs entières. Déterminer la condition pour que le rayon du cercle inscrit du triangle $ABC$ soit un nombre rationnel.

**M499**. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Deux cercles de rayon $1$ sont tracés de façon à ce que chacun passe par le centre de l'autre. Déterminer la surface de la région en forme de gobelet qui se trouve entre le rayon commun, les circonférences et une des tangentes communes, tel qu'illustré à droite.



**M500**. *Proposé par Edward T.H. Wang et Dexter S.Y. Wei, Université Wilfrid Laurier, Waterloo, ON.*

Soit $\mathbb{N}$ l'ensemble des nombres naturels.

(a) Démontrer que si $n \in \mathbb{N}$ alors il n'existe aucun $a, b \in \mathbb{N}$ tels que $\dfrac{[a,b]}{a+b} = n$, où $[a, b]$ dénote le plus petit commun multiple de $a$ et $b$.

(b) Démontrer que si $n \in \mathbb{N}$ alors il existe un nombre infini de triplets $(a, b, c)$ d'entiers naturels tels que $\dfrac{[a,b,c]}{a+b+c} = n$, où $[a, b, c]$ dénote le plus petit commun multiple de $a$, $b$ et $c$.

# Mayhem Solutions

**M457**. *Proposed by the Mayhem Staff.*

Suppose that $A$ is a digit between $0$ and $9$, inclusive, and that the tens digit of the product of $2A7$ and $39$ is $9$. Determine the digit $A$.

*Solution by Florencio Cano Vargas, Inca, Spain.*

We write $2A7 = 2 \cdot 10^2 + A \cdot 10 + 7$ and $39 = 3 \cdot 10 + 9$. Multiplying both numbers and grouping we get:

$$2A7 \cdot 39 = 8 \cdot 10^3 + 3A \cdot 10^2 + (9A + 7) \cdot 10 + 3\,.$$

The condition stated in the problem implies that $9A + 7 \equiv 9 \pmod{10}$ which implies that $9A \equiv 2 \pmod{10}$. Hence, the solution is $A = 8$.

*Also solved by JACLYN CHANG, student, University of Calgary, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; LUIZ ERNESTO LEITÃO, Pará, Brazil; TRAVIS B. LITTLE, students, Angelo State University, San Angelo, TX, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; INGESTI BILKIS ZULPATINA, student, SMPN 8, Yogyakarta, Indonesia;*

**M458**. *Proposed by the Mayhem Staff.*

Convex quadrilateral $ABCD$ has $AB = AD = 10$ and $BC = CD$. Also, $AC$ is perpendicular to $BD$, with $AC$ and $BD$ intersecting at $P$. If $BP = 8$ and $CD = CP + 2$, determine the area of quadrilateral $ABCD$.

*Solution by Ingesti Bilkis Zulpatina, student, SMPN 8, Yogyakarta, Indonesia.*

From the properties which are written above, $ABCD$ is surely a kite since $AB = AD$, $BC = CD$, and $AC \perp BD$.