$\lfloor x \rfloor \lceil x \rceil = n(n+1) = x^2$, hence $x = -\sqrt{n(n+1)}$. Thus, the set of all real numbers for which $\lfloor x \rfloor \lceil x \rceil = x^2$ is $x = \pm n$ or $x = \pm\sqrt{n(n+1)}$, $n \in \mathbb{N} \cup \{0\}$.

Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and the proposers. Three incomplete solutions were received.

Problem of the Month

Ian VanderBurgh

Many problems that appear on contests are word problems. Here is a problem that appeared last year on a Scottish competition:

Problem (2010-2011 Scottish Mathematical Challenge) Katie had a collection of red, green and blue beads. She noticed that the number of beads of each colour was a prime number and that the numbers were all different. She also observed that if she multiplied the number of red beads by the total number of red and green beads she obtained a number exactly 120 greater than the number of blue beads. How many beads of each colour did she have?

Often, the first step with a word problem is to translate the words into mathematics. Since this problem is dealing with the numbers of red, green and blue beads, let's assign a variable to each of these numbers – say, r, g and b, respectively. (We'll write this up nicely in a minute.) These seem to be the relevant quantities.

We are next told that each of these quantities is a prime number. Let's make a mental note to come back to this, and keep reading. The fact that the product of the number of red beads with the sum of the numbers of red and green beads is 120 more than the number of blue beads translates into the equation $r(r+g) = 120 + b$.

Now, I seem to remember that usually when we have three variables, one equation is not enough to determine the values of the variables. (Often, we need three equations.) This is mildly concerning, but let's persevere to see what happens.

What information haven't we used? We haven't used the fact that each of r, g and b is a prime number. How can we use this information? Again, let's back up half a step. What do we know about prime numbers? It's good to check the definition first: a prime number is a positive integer larger than 1 (remember, 1 is not prime) that has no positive divisors other than 1 and itself. Is there a "formula" for prime numbers? There isn't a good one that we know. However, there are lots and lots of properties of prime numbers: all prime numbers other than 2 are odd, there are infinitely many prime numbers, every prime number greater than $\mathbf{3}$ is either one more or one less than a multiple of $\mathbf{6}$... The list goes on! Many mathematicians spend much of their professional lives investigating properties of prime numbers.

Given such a vast number to choose from, how do we know what properties to use? Therein lies the essence of problem solving! Figuring this out is not always easy.

Here's a solution to the problem.

Solution Suppose that $\boldsymbol{r}$, $\boldsymbol{g}$ and $\boldsymbol{b}$ are the numbers of red, green and blue beads, respectively. We are told that each of $\boldsymbol{r}$, $\boldsymbol{g}$ and $\boldsymbol{b}$ is a prime number and that $\boldsymbol{r(r+g)=120+b}$.

Let's focus on the fact that the only even prime number is $\mathbf{2}$ and on the parity of the two sides of the equation. (Remember, checking parity means checking to see if an integer is even or odd.) If both $\boldsymbol{r}$ and $\boldsymbol{g}$ are odd, then $\boldsymbol{r+g}$ is even, so the left side of the equation is even, which means the right side is even. If $\boldsymbol{120+b}$ is even, then $\boldsymbol{b}$ is even, which means that $\boldsymbol{b=2}$. In this case, $\boldsymbol{r(r+g)=122}$. Since $\mathbf{122=2\times 61}$ and each of $\mathbf{2}$ and $\mathbf{61}$ is prime, then we must have $\boldsymbol{r=2}$ or $\boldsymbol{r+g=2}$. Neither of these is possible, since $\boldsymbol{r}$ cannot equal $\boldsymbol{b}$ (since $\boldsymbol{r}$, $\boldsymbol{b}$ and $\boldsymbol{g}$ are all different) and since $\boldsymbol{r+g}$ is at least $\mathbf{4}$.

Also, since $\mathbf{2}$ is the only even prime number, then $\boldsymbol{r}$ and $\boldsymbol{g}$ can't both be even, since then they would both be $\mathbf{2}$, which would contradict the given hypothesis that $\boldsymbol{r}$, $\boldsymbol{b}$ and $\boldsymbol{g}$ are all different.

Therefore, $\boldsymbol{r}$ and $\boldsymbol{g}$ are even and odd in some order. In other words, one of $\boldsymbol{r}$ and $\boldsymbol{g}$ equals 2 and the other is an odd prime number. Which is which? If $\boldsymbol{r=2}$, then the equation becomes $\boldsymbol{2(2+g)=120+b}$. Since the left side is even, then the right side is even too, so again $\boldsymbol{b=2}$ which is impossible since our assumption is that $\boldsymbol{r=2}$.

Thus, it must be the case that $\boldsymbol{g=2}$ and $\boldsymbol{r}$ is an odd prime. This gives $\boldsymbol{r(r+2)=120+b}$.

So we've got one value ($\boldsymbol{g=2}$), but still have one equation and two unknowns. What to do?

Let's try solving for $\boldsymbol{b}$, which gives $\boldsymbol{b=r^2+2r-120}$. At this point, it might occur to try to factor the right side to obtain $\boldsymbol{b=(r+12)(r-10)}$.

How does this help? Since $\boldsymbol{b}$ is a prime number, then it can't be factored in many ways! Aha – that is probably useful. If $\boldsymbol{b}$ is a prime number that is written as the product of two integers, then one of the factors is either $\mathbf{1}$ or $\mathbf{-1}$. This gives us four possibilities to check ($\boldsymbol{r+12}$ equals $\mathbf{1}$ or $\mathbf{-1}$ and $\boldsymbol{r-10}$ equals $\mathbf{1}$ or $\mathbf{-1}$). The only one that yields a positive value of $\boldsymbol{r}$ that is a prime number is $\boldsymbol{r-10=1}$, giving $\boldsymbol{r=11}$. In this case, $\boldsymbol{b=(11+12)(11-10)=23}$, which is (thankfully) a prime number.

Therefore, $\boldsymbol{g=2}$, $\boldsymbol{r=11}$ and $\boldsymbol{g=23}$. We can check that these satisfy the original hypotheses. □