SKOLIAD No. 134

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Please send your solutions to problems in this Skoliad by **July 15, 2012**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Baden-Württemberg Mathematics Contest, 2010. Our thanks go to the Landeswettbewerb Mathematik Baden Württemberg for providing us with this contest and for permission to publish it. We also thank Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the contest.

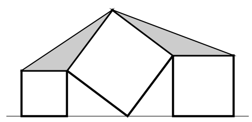
Baden-Württemberg Mathematics Contest, 2010

1. Sonja has nine cards on which the nine smallest two-digit prime numbers are printed. She wants to order these cards in such a way that neighbouring cards always differ by a power of $\mathbf{2}$. In how many ways can Sonja order her cards?

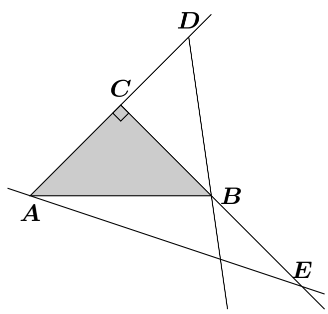
2. A $\mathbf{50}$ cm by $\mathbf{30}$ cm by $\mathbf{28}$ cm box contains wooden blocks that all measure $\mathbf{10}$ cm by $\mathbf{9}$ cm by $\mathbf{7}$ cm. At most how many blocks can fit in the box? Explain how to fit that many blocks into the box.

3. Five distinct positive numbers are given. Forming all possible sums of two of these numbers you obtain seven different sums. Show that the sum of the five original numbers is divisible by $\mathbf{5}$.

4. Three squares are arranged as in the figure. Show that the two shaded triangles have the same area.



5. Triangle $\triangle \boldsymbol{ABC}$ is isosceles and $\angle \boldsymbol{ACB} = \mathbf{90}^\circ$. The point $\boldsymbol{D}$ is on the line $\boldsymbol{AC}$ beyond $\boldsymbol{C}$, and the point $\boldsymbol{E}$ is on the line $\boldsymbol{CB}$ beyond $\boldsymbol{B}$. Show that $|\boldsymbol{CD}| = |\boldsymbol{CE}|$ if line $\boldsymbol{BD}$ is perpendicular to line $\boldsymbol{AE}$.



6. The product of three positive integers is three times as large as their sum. Find all such triples.

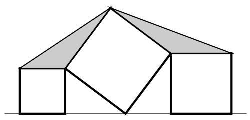
Concours mathématique Baden-Württemberg 2010

1. Sonya dispose de neuf cartes sur lesquelles sont imprimés les neuf plus petits nombres premiers à deux chiffres. Elle voudrait ordonner ses cartes de façon à ce que les cartes voisinantes diffèrent toujours par une puissance de **2**. De combien de manières est-ce que Sonya peut ordonner ses cartes ?

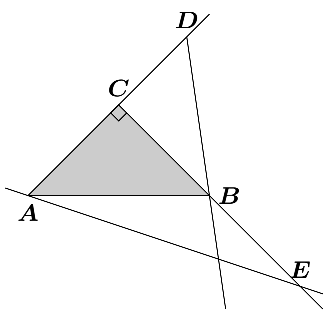
2. Une boîte de taille **50** cm par **30** cm par **28** cm contient des blocs en bois, chacun de taille **10** cm par **9** cm par **7** cm. Au plus, combien de tels blocs entrent dans la boîte ? Expliquer comment faire entrer ce nombre de blocs dans la boîte.

3. Cinq nombres positifs distincts vous sont donnés. En formant toutes les sommes possibles de deux de ces nombres, on constate qu'on obtient sept sommes distinctes. Démontrer que la somme des cinq nombres originaux est divisible par **5**.

4. Trois carrés sont disposés tel qu'illustré par la figure. Démontrer que les deux triangles ombragés ont la même surface.



5. Le triangle $\triangle ABC$ est isocèle et $\angle ACB = 90°$. Le point D se trouve sur la ligne AC, au-delà de C, et le point E se trouve sur la ligne CB, au-delà de B. Démontrer que $|CD| = |CE|$ si la ligne BD est perpendiculaire à la ligne AE.



6. Le produit de trois entiers positifs est trois fois aussi grand que leur somme. Déterminer tout tel triplet.

Next follow solutions to the Mathematics Association of Quebec Contest, Secondary level, 2010, given in Skoliad 128 at [2010:417–419].

1. An *alphametic* is a small mathematical puzzle consisting of an equation in which the digits have been replaced by letters. The task is to identify the value of each letter in such a way that the equation comes out true. Different letters have different values, different digits are represented by different letters, and no number begins with a zero. For example, the alphametic PAPA + PAPA = MAMAN has the solution P = **7**, A = **5**, M = **1**, and N = **0**, yielding **7575 + 7575 = 15150**.

Find the solution to this "reversing" alphametic:

$$\mathrm{NOMBRE} \times \frac{\mathbf{3}}{\mathbf{5}} = \mathrm{ERBMON}\,.$$

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

If NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ = ERBMON, then NOMBRE × **3** = ERBMON × **5**, so NOMBRE is divisible by **5**. Therefore E = **0** or E = **5**. However, ERBMON does not begin with a zero, so E = **5**.

Thus ERBMON > **500 000**, so NOMBRE = $\frac{\mathbf{5}}{\mathbf{3}}$ × ERBMON > **833 333**. Therefore N = **8** or N = **9**. When multiplying integers, the ones digit of the result depends only on the ones digits of the factors. Therefore, the ones digit of NOMBRE × **3** is the ones digit of E × **3** which is **5**, since E = **5**. Now, NOMBRE × **3** = ERBMON × **5**, so the ones digit of ERBMON × **5** is also **5**. Therefore, the ones digit of N × **5** must be **5**, so N cannot be **8**. Thus N = **9**.

You now know that the ones digit of NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ is **9**. Note that the ones digit of NOMBRE ÷ **5** equals the ones digit of RE ÷ **5**. Therefore the ones digit of NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ equals the ones digit of RE $\times \frac{\mathbf{3}}{\mathbf{5}}$, which, thus, must be **9**. Since E = **5**, either R = **1** or R = **6**. If R = **1**, then NOMBRE = ERBMON $\times \frac{\mathbf{5}}{\mathbf{3}} \leq$ **519 999** $\times \frac{\mathbf{5}}{\mathbf{3}}$ = **866 665**, contradicting the fact that N = **9**. Thus R = **6**.

Now, NOMBRE = ERBMON $\times \frac{\mathbf{5}}{\mathbf{3}} \geq$ **560 000** $\times \frac{\mathbf{5}}{\mathbf{3}}$ > **933 333**, so O ≥ **3**. Moreover, ERBMON is divisible by **3**, so E + R + B + M + O + N is divisible by **3**. Thus NOMBRE is divisible by **3**, and, hence, ERBMON = NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ is divisible by **9**. Therefore, N + O + M + B + R + E = **9** + O + M + B + **6** + **5** = O + M + B + **20** is divisible by **9**.

You must now find three digits, (O, M, B), from $\{\mathbf{0,1,2,3,4,7,8}\}$ subject to the two conditions that O ≥ **3** and that O + M + B + **20** is divisible by **9**. Only ten triples satisfy these conditions: **(3, 0, 4)**, **(3, 4, 0)**, **(4, 0, 3)**, **(4, 1, 2)**, **(4, 2, 1)**, **(4, 3, 0)**, **(7, 1, 8)**, **(7, 8, 1)**, **(8, 1, 7)**, and **(8, 7, 1)**. Of these only one satisfy the further condition that NOMBRE $\times \frac{\mathbf{3}}{\mathbf{5}}$ = ERBMON, namely (O, M, B) = **(3, 4, 0)**.

$$\mathbf{934\,065} \times \frac{\mathbf{3}}{\mathbf{5}} = \mathbf{560\,439}$$

The solution **934 065** $\times \frac{\mathbf{3}}{\mathbf{5}}$ = **560 439** *was also found by ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; JANICE LEW, student, École Alpha*

Secondary School, Burnaby, BC; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

2. Find all polynomials of the form $p(x) = x^3+mx+6$ whose roots are integers.

Solution by Billy Suandito, Palembang, Indonesia.

All integer roots of $p(x)$ must be factors of 6 (*for the reason why, see the editors' note below*). Thus, the possible integer roots are 1, 2, 3, 6, -1, -2, -3, and -6.

If 1 is a root, then $0 = p(1) = 1^3 + m + 6 = 7 + m$, so $m = -7$. Thus $p(x) = x^3 - 7x + 6 = (x-1)(x^2+x-6) = (x-1)(x+3)(x-2)$. Thus $p(x)$ has three integer roots if $m = -7$.

If 2 (or -3) is a root, you find the above example again.

If 3 is a root, then $0 = p(3) = 3^3 + 3m + 6 = 33 + 3m$, so $m = -11$. Thus $p(x) = x^3 - 11x + 6 = (x-3)(x^2+3x-2)$, but the roots of x^2+3x-2 are not integers.

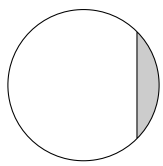
If you continue checking the possible roots, 6, -1, -2, and -6, in a similar manner, you will find that $p(x)$ fails to have integer roots in each case except $m = -7$. Thus $p(x) = x^3 - 7x + 6$ is the only solution.

The solution $p(x) = x^3 - 7x + 6$ was also found by WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

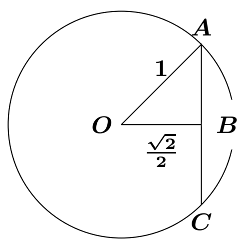
Alternatively, say the three roots are a, b, and c. Then $x^3 + mx + 6 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$. Thus $a+b+c = 0$ and $abc = -6$. It follows that exactly one of the roots is negative. Since the roots are integers, they must be factors of 6. The only possibility is, then, that the roots are 1, 2, and -3. Thus the only solution is $p(x) = (x-1)(x-2)(x+3) = x^3 - 7x + 6$ as above.

Note, as in the previous paragraph, that if the leading coefficient of a polynomial is 1, then the next-to-leading coefficient is the negative of the sum of the roots and the constant term is, apart from a sign, the product of the roots. This observation often comes in handy in contests.

3. A line is located at $\frac{\sqrt{2}}{2}$ units from the centre of a circle of radius 1, separating it into two parts. What is the area of the smaller part?



Solution by Lisa Wang, student, Port Moody Secondary School, Port Moody, BC.



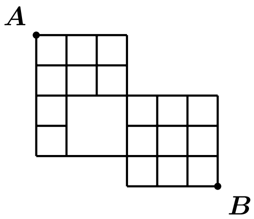
Let O be the centre of the circle and A and C be the endpoints of the chord. Let B be the point on AC that is closest to O. Then $|OB| = \frac{\sqrt{2}}{2}$, $\angle OBA = 90^\circ$, and B is the midpoint of AC. By the Pythagorean Theorem, $|AB| = \sqrt{1^2 - (\frac{\sqrt{2}}{2})^2} = \sqrt{1 - \frac{1}{2}} = \frac{\sqrt{2}}{2}$. Therefore $|AC| = \sqrt{2}$, so the area of $\triangle AOC$ is $\frac{1}{2} \cdot |AC| \cdot |OB| = \frac{1}{2} \cdot \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$.

Moreover, $\triangle AOB$ is isosceles and $\angle AOB = 45^\circ$. By symmetry, $\angle COB = 45^\circ$, so $\angle AOC = 90^\circ$, whence sector AOC is a quarter circle. Thus the area of sector AOC is $\frac{1}{4}\pi 1^2 = \frac{\pi}{4}$.

It follows that the area of the shaded segment is $\frac{\pi}{4} - \frac{1}{2} \approx \mathbf{0.285}$.

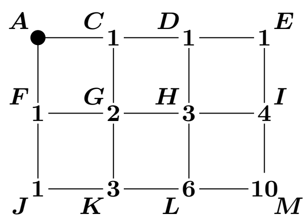
Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; VINCENT CHUNG, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and BILLY SUANDITO, Palembang, Indonesia.

4. The figure shows a map of a city. In how many ways can you travel along the roads of the city from point A to point B if you can only travel east and south (right and down in the figure)?

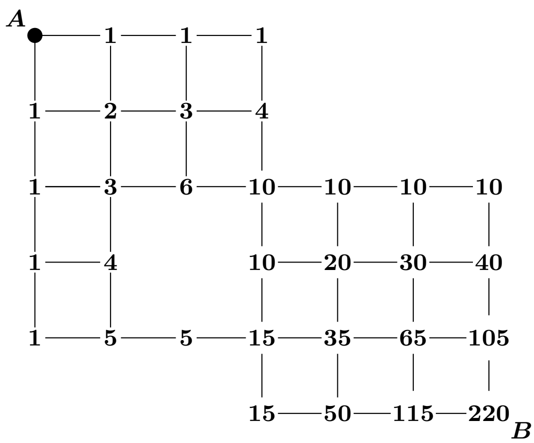


Solution by Vincent Chung, student, Burnaby North Secondary School, Burnaby, BC.

Consider this simpler map. Surely, you can arrive at C in only one way, as indicated. To reach D, you must come from C, so you can arrive at D in only one way. Similarly at E and F. You can reach G either from C or from F, so you can arrive at G in two ways. You can reach H either from D or from G, so you can arrive at H in $1 + 2 = 3$ ways. Now continue in this way: if you can reach X from Y and Z, then the number of paths to X is the sum of the number of paths to Y and the number of paths to Z.



Using this method of counting paths in the original map yields a total of **220** paths from A to B:



Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; JANICE LEW, student, École Alpha Secondary School, Burnaby, BC; and BILLY SUANDITO, Palembang, Indonesia.

5. (a) How many zeroes are at the right-hand end of the number $\mathbf{1 \times 2 \times 3 \times \cdots \times 52}$?

Solution by Janice Lew, student, École Alpha Secondary School, Burnaby, BC.

The number $\mathbf{1 \times 2 \times \cdots \times 52}$ is written more compactly as $\mathbf{52!}$. Let $\boldsymbol{S}$ denote the set $\mathbf{\{1, 2, 3, \ldots, 52\}}$. Half of the numbers in $\boldsymbol{S}$ are even, so $\mathbf{52!}$ is divisible by $\mathbf{2^{26}}$. One quarter of the numbers in $\boldsymbol{S}$ are divisible by $\mathbf{4}$. These contribute an extra $\mathbf{\frac{52}{4} = 13}$ copies of $\mathbf{2}$. Since $\mathbf{\frac{52}{8} = 6.5}$, the numbers in $\boldsymbol{S}$ that are divisible by $\mathbf{8}$ contribute a further $\mathbf{6}$ copies of $\mathbf{2}$. Three of the numbers in $\boldsymbol{S}$ are divisible by $\mathbf{16}$, and one is divisible by $\mathbf{32}$. Thus $\mathbf{52!}$ is divisible by $\mathbf{2^{26+13+6+3+1} = 2^{49}}$ but not by any higher power of $\mathbf{2}$.

Likewise, $\mathbf{52!}$ is divisible by $\mathbf{5^{10+2} = 5^{12}}$ but no higher power of $\mathbf{5}$. Since $\mathbf{10 = 2 \times 5}$, it follows that $\mathbf{52!}$ is divisible by $\mathbf{10^{12}}$ but no higher power of $\mathbf{10}$. Thus $\mathbf{52!}$ ends in exactly $\mathbf{12}$ zeroes.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; VINCENT CHUNG, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

(b) What is the rightmost nonzero digit of $\mathbf{1 \times 2 \times \cdots \times 52}$? (For example, the rightmost nonzero digit of $\mathbf{1 \times 2 \times \cdots \times 12 = 479\,001\,600}$ is $\mathbf{6}$.)

Solution by the editors.

In Part (a), our solver began factoring $\mathbf{52!}$ into primes. Now complete the process: Since $\mathbf{\frac{52}{3} \approx 17.3}$, $\mathbf{\frac{52}{9} \approx 5.8}$, and $\mathbf{\frac{52}{27} \approx 1.9}$, it follows that $\mathbf{52!}$ is divisible by $\mathbf{3^{17+5+1} = 3^{23}}$. Likewise, $\mathbf{52!}$ is divisible by $\mathbf{7^{7+1} = 7^8}$, $\mathbf{11^4}$, $\mathbf{13^4}$, $\mathbf{17^3}$, $\mathbf{19^2}$, $\mathbf{23^2}$, and once each by $\mathbf{29}$, $\mathbf{31}$, $\mathbf{37}$, $\mathbf{41}$, $\mathbf{43}$, and $\mathbf{47}$. Thus $\mathbf{52! = 2^{49} \cdot 3^{23} \cdot 5^{12} \cdot 7^8 \cdot 11^4 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47}$.

The rightmost non-zero digit of $\mathbf{52!}$ equals the ones digit of

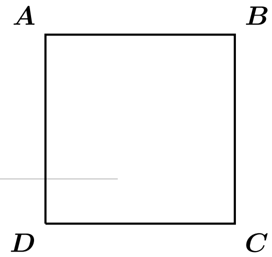
$$\mathbf{\frac{52!}{10^{12}} = 2^{37} \cdot 3^{23} \cdot 7^8 \cdot 11^4 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47}\,.$$

Let $\boldsymbol{a \equiv b}$ denote that the whole numbers $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same ones digit. Since the ones digit of a product depends only on the ones digits of the factors, $\mathbf{2^{37} = 2^2 \cdot (2^5)^7 = 2^2 \cdot (32)^7 \equiv 2^2 \cdot 2^7 = 2^9 = 512 \equiv 2}$. Likewise, $\mathbf{3^{23} = 3^3 \cdot (3^4)^5 = 3^3 \cdot (81)^5 \equiv 3^3 \cdot 1 = 27 \equiv 7}$, and $\mathbf{7^8 = (7^4)^2 = (2401)^2 \equiv 1}$, and $\mathbf{11^4 \equiv 1}$, and $\mathbf{13^4 \equiv 3^4 = 81 \equiv 1}$, and $\mathbf{17^3 \equiv 7^3 = 343 \equiv 3}$, and $\mathbf{19^2 \equiv 9^2 = 81 \equiv 1}$, and $\mathbf{23^2 \equiv 3^2 = 9}$, and, of course, $\mathbf{29 \equiv 9}$, $\mathbf{31 \equiv 1}$, $\mathbf{37 \equiv 7}$, $\mathbf{41 \equiv 1}$, $\mathbf{43 \equiv 3}$, and $\mathbf{47 \equiv 7}$. Thus

$$\mathbf{\frac{52!}{10^{12}} \equiv 2 \cdot 7 \cdot 1 \cdot 1 \cdot 1 \cdot 3 \cdot 1 \cdot 9 \cdot 9 \cdot 1 \cdot 7 \cdot 1 \cdot 3 \cdot 7 = 500094 \equiv 4}\,.$$

Therefore the rightmost non-zero digit of $\mathbf{52!}$ is $\mathbf{4}$.

6. Juliette and Philippe play the following game: At the beginning of the game, each corner of a square is covered with a number of chips. In turn, each player chooses one side of the square and removes as many chips as (s)he wants from the endpoints of that side provided (s)he takes at least one chip. It is not necessary to remove the same number of chips from each endpoint. The player who removes the last chip wins.



At the beginning of the game on the square $ABCD$ there are 10 chips on corner A, 11 chips on B, 12 chips on C, and 13 chips on D. If Juliette begins, how should she play?

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC; Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and Szera Pinter, student, Moscrop Secondary School, Burnaby, BC.

Let a, b, c and d denote the number of chips on A, B, C, and D, respectively. Strategy: Juliette should play to ensure that $a = c$ and $b = d$.

If Philippe receives a square with $a = c$ and $b = d$, then he must remove at least one chip, and he cannot remove chips from both ends of a diagonal. Therefore he will always pass a square to Juliette with $a \neq c$ and/or $b \neq d$.

If Juliette receives a square with $a \neq c$ and/or $b \neq d$, then for each diagonal she should choose the endpoint with the larger number of chips. Then she should choose the side that connects those two endpoints and remove chips until $a = c$ and $b = d$.

Thus Juliette is always able to follow the strategy above. Eventually, $a = c = 0$ and $b = d = 0$, and Juliette wins.

7. Find all functions $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) + xF(-x) = 1$ for all real numbers x.

Solution by Billy Suandito, Palembang, Indonesia.

If $F(x) + xF(-x) = 1$ for *all* values of x, then the equation also holds for $-x$; that is, $F(-x) - xF(x) = 1$. Multiplying this last equation by x yields that $xF(-x) - x^2F(x) = x$. Subtract this from the original equation to get that $F(x) + x^2F(x) = 1 - x$, so $(1 + x^2)F(x) = 1 - x$, so $F(x) = \dfrac{1-x}{1+x^2}$.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

We invite the reader to submit solutions to one or more of our problems.