RECURRING CRUX CONFIGURATIONS

J. Chris Fisher

Triangles for which $2b^2 = c^2 + a^2$

Many configurations have popped up again and again in **Crux** geometry problems over the years. In this occasionally appearing column we will recall some of them—they might not be as common as a right triangle, but their properties are generally useful and often surprising. This month we investigate triangles with sides a, b, c that satisfy $2b^2 = c^2 + a^2$. Following the suggestion of Leon Bankoff we will call them **root-mean-square triangles** because $b = \sqrt{\frac{c^2+a^2}{2}}$ is the root mean square of a and c. These triangles have also been called *automedian* and *self-median* (see property 1 below), as well as *quasi-isosceles* (see problem 727 below).

Leon Bankoff reported [1978 : 13-16] that in a series of *Mathesis* articles [4], 69 properties of these triangles are listed, generally with an abundance of earlier references instead of proofs. He listed his favorite 10 properties and supplied the short proofs. Here we shall leave the proofs as exercises with hints where appropriate. For these ten properties we assume that ABC is a triangle whose sides satisfy

$$2b^2 = c^2 + a^2. \tag{1}$$

Let m_a, m_b, m_c denote the medians to sides a, b, c, respectively, and let H, O, and G denote the orthocentre, circumcentre, and centroid. The symmedians k_a, k_b, and k_c are the reflections of the medians in the respective angle bisectors; they meet in the symmedian point K. As usual, let $s = \frac{a+b+c}{2}$ be the semiperimeter, and R denote the circumradius.

Property 1. $m_a = \frac{\sqrt{3}}{2}c,\ m_b = \frac{\sqrt{3}}{2}b,\ m_c = \frac{\sqrt{3}}{2}a$.

This property follows immediately by plugging (1) into the formula for the length of a median in terms of the sides. Note that the medians can be translated to form a triangle that is oppositely similar to triangle ABC; this property is the source of the "automedian" terminology that was used throughout [4]. Perhaps in French the word sounds less asinine, but to me it does not convey the image of a triangle that is oppositely similar to the triangle formed by its three medians. K.R.S. Sastry favored the terminology self-median. He proved a converse in "The Triangle: A Parametric Description" [2004 : 497-501]: *If b is the length of the middle side, then ΔABC is (oppositely) similar to its medial triangle if and only if $b = \frac{\sqrt{3}}{2}m_b$, if and only if (1) holds.* Note that the ratio of similitude of corresponding sides tells us that the area of the triangle formed by the medians is $3/4$ that of the original triangle. It is also seen that $m_a + m_b + m_c = \sqrt{3}s$, a familiar property of the equilateral triangle, which is, of course, a special case of a root-mean-square triangle.

Property 2. $b^2 = 2ca\cos B.$ (This is the cosine law applied to (1).)

Property 3. $\mathbf{2\cot B = \cot C + \cot A}$. (You can get started by applying the sine law to both sides of (1).)

Property 4. $\mathbf{2\cos 2B = \cos 2C + \cos 2A}$.

Property 5. $b^2 = AG^2 + BG^2 + CG^2$.

Property 6. $4\cdot\text{Area}(ABC) = b^2 \tan B$.

Property 7. $2m_b^2 = m_c^2 + m_a^2$.

Property 8. $2BH^2 = CH^2 + AH^2$.

Property 9. *If B' and B'' are the third vertices of equilateral triangles constructed externally and internally on side AC, then $\angle B'BB''$ is a right angle.*

Hint. A 1796 theorem of Nicholas Fuss implies that $B'B^2 + B''B^2 = a^2 + b^2 + c^2$ [3, p. 220, par. 354c]; more easily than tracking down a reference, one can prove Fuss's theorem by applying the cosine law to triangles BCB' and BCB''.

Property 10. $\frac{m_b}{k_b} = 2\cos B$.

Also in [1978 : 13], W.J. Blundon showed how to construct a triangle satisfying (1) given the segment AC: Construct the equilateral triangle ACD and let O be the midpoint of AC; Property 1 tells us that a triangle ABC satisfies (1) if and only if $OB = \frac{\sqrt{3}}{2}b$, whence the locus of B is the circle with centre O and radius OD. The editor Léo Sauvé added (on page 16) that if you want to generate specific examples of root-mean-square triangles with integer sides, you could use the theorem that all primitive solutions of $2b^2 = c^2 + a^2$ are given by

$$a = u^2 + 2uv - v^2, \quad b = u^2 + v^2, \quad c = |u^2 - 2uv - v^2|,$$

where $u > v$, with u and v relatively prime positive integers of different parity [1], [2, pp. 435 ff.]. Of course, your values of a, b, c must satisfy the triangle inequality. Another approach to this result was taken by K.R.S. Sastry in "Pythagoras Strikes Again!" [1998 : 276-280]; specifically, he proved that

> If a_0, b_0, c_0 are the sides of a right triangle in which $c_0^2 + a_0^2 = b_0^2$, $a_0 > c_0$, and $b_0 > 2c_0$, then $a = a_0 + c_0$, $c = a_0 - c_0$, and $b = b_0$ are the sides of a triangle that satisfies $c^2 + a^2 = 2b^2$; conversely, if the sides of a triangle satisfy $a > b > c$ and $c^2 + a^2 = 2b^2$, then $a_0 = \frac{1}{2}(a + c)$ and $c_0 = \frac{1}{2}(a - c)$ are the legs of a right triangle whose hypotenuse is b, $a_0 > c_0$, and $b > 2c_0$.

For a list of all $\mathbf{36}$ primitive root-mean-square triangles with perimeters less than 1000, see [1978 : 194]. The smallest (that is not equilateral) has side lengths $\mathbf{17, 13, 7}$, which comes from a 12-13-5 right triangle by Sastry's theorem.

The 1978 discussion of these triangles came out of the solution to problem 210 [1977 : 10, 160-164, 196-197; 1978 : 13-16, 193-194] (proposed by Murray S. Klamkin): P, Q, R denote points on the sides BC, CA, and AB, respectively, of a given triangle ABC; determine all triangles ABC such that if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \qquad \left(\neq 0, \frac{1}{2}, 1\right),$$

then $\boldsymbol{PQR}$ (in some order) is similar to $\boldsymbol{ABC}$. The solution revealed that if the triangles are directly similar, then they must be equilateral. When $\boldsymbol{a > b > c}$ the value of $\boldsymbol{k}$ in one of the three resulting solutions has $\boldsymbol{2b^2 - c^2 - a^2}$ in its denominator, whence *root-mean-square triangles* $\boldsymbol{ABC}$ *would have only two oppositely similar triangles* $\boldsymbol{PQR}$ *instead of three, and those two triangles would be congruent.*

Problem 309 [1978 : 12, 200-202] (Proposed by Peter Shor). Let $\boldsymbol{ABC}$ be a triangle with $\boldsymbol{a \geq b \geq c}$ or $\boldsymbol{a \leq b \leq c}$. Let the bisectors of $\angle \boldsymbol{cm_a}$ and $\angle \boldsymbol{am_c}$ meet at $\boldsymbol{R}$. Prove that

(a) $\boldsymbol{AR \perp CR}$ if and only if $\boldsymbol{2b^2 = c^2 + a^2}$;

(b) if $\boldsymbol{2b^2 = c^2 + a^2}$, then $\boldsymbol{R}$ lies on $\boldsymbol{m_b}$.

Is the converse of (b) true?

Yes, the converse turns out to be true (when $\boldsymbol{b}$ is assumed to be the middle side). The featured solution of Daniel Sokolowsky made use of yet another interesting property:

Theorem. If $\boldsymbol{D}$ and $\boldsymbol{E}$ are the midpoints of sides $\boldsymbol{AB}$ and $\boldsymbol{BC}$, and $\boldsymbol{G}$ is the centroid, then $\boldsymbol{2b^2 = c^2 + a^2}$ if and only if $\boldsymbol{BDGE}$ is a cyclic quadrilateral.

Problem 313 (revised) [1978 : 35, 207-209] (Proposed by Leon Bankoff). The sides of a nonequilateral triangle satisfy $\boldsymbol{2b^2 = c^2 + a^2}$ if and only if $\boldsymbol{GK}$ (the join of the centroid and the symmedian point) is parallel to $\boldsymbol{AC}$.

Bankoff seems not to have worried much about converses! Although his version of Problem 313 called for a proof of sufficiency only, both featured solutions make clear that (1) is both necessary and sufficient for $\boldsymbol{GK}$ to be parallel to $\boldsymbol{AC}$. He found the theorem (without a converse) in *Mathesis*, t. IX (1889) p. 208, where it was attributed to Lemoine (*Mathesis*, t. V (1885) p. 104). As for Bankoff's list of ten properties (reproduced above), although I did not carefully write down the proofs, I believe that all ten converses hold for triangles with $\boldsymbol{a > b > c}$ or $\boldsymbol{c > b > a}$.

Problem 383 [1978 : 250, 174-176] (Proposed by Daniel Sokolowsky). Let $\boldsymbol{m_a, m_b, m_c}$ be respectively the medians $\boldsymbol{AD, BE, CF}$ of a triangle $\boldsymbol{ABC}$ with centroid $\boldsymbol{G}$. Prove that

(a) if $\boldsymbol{m_a : m_b : m_c = a : b : c}$, then $\boldsymbol{\Delta ABC}$ is equilateral;

(b) if $\boldsymbol{\frac{m_b}{m_c} = \frac{c}{b}}$, then either (i) $\boldsymbol{b = c}$ or (ii) quadrilateral $\boldsymbol{AEGF}$ is cyclic;

(c) if both (i) and (ii) hold in (b), then $\boldsymbol{\Delta ABC}$ is equilateral.

By the theorem from the proof of Problem 309 above, quadrilateral $\boldsymbol{AEGF}$ is cyclic in part (b) if and only if $\boldsymbol{2a^2 = b^2 + c^2}$. Part (b) should be compared with the quasi-isosceles property of Bottema and Groenman discussed in the next problem, number 727.

Problem 727 [1982 : 78; 1983 : 115, 180-181] (Proposed by J.T. Groenman). Let $\boldsymbol{t_b}$ and $\boldsymbol{t_c}$ be the symmedians issued from vertices $\boldsymbol{B}$ and $\boldsymbol{C}$ of triangle $\boldsymbol{ABC}$ and terminating in the opposite sides $\boldsymbol{b}$ and $\boldsymbol{c}$, respectively. Prove that $\boldsymbol{t_b = t_c}$ if and only if $\boldsymbol{b = c}$.

It turned out that this problem had already appeared several times elsewhere; instead of a solution, three references were provided: *Amer. Math.*

Monthly, **51** (Dec 1944) 590-591; *Scripta Mathematica*, **22** (1956) 102; and *Mathematics Magazine*, **40** (May 1967) 165, and **41** (Jan 1968) 48-49. There was also reference to a problem in the *Pi Mu Epsilon Journal* that claimed to prove that the analogous result holds also for exsymmedians. Not so, according to a paper by Groenman with O. Bottema in *Nieuw Tijdschrift voor Wiskunde*, **70** (1983) 143-151. For the details, recall that an exmedian is a line through a vertex parallel to the opposite side of the triangle, while an exsymmedian is the reflection of the exmedian in the external angle bisector. The Bottema-Groenman result says that if the exsymmedians from B and C are equal in length, then either $b = c$ (and ΔABC is isosceles), or $2a^2 = b^2 + c^2$ (and the triangle is a root-mean-square triangle). In the latter case they call the triangle *quasi-isosceles.* Warning: their article is written in Dutch.

After Problem 727 (from 28 years ago!) it seems as if Sastry has single-handedly kept the subject of root-mean-square triangles alive. Besides the **Crux** articles from 1998 and 2004 mentioned earlier, he published the article [5] and posed Problem 473 in [6]: *Prove that in a scalene triangle* ABC*, the bisectors of* $\angle ABC$ *and* $\angle AGC$ *intersect on the side* AC *if and only if* $c^2 + a^2 = 2b^2$. He is also responsible for

Problem 2252 [1997 : 300; 1998 : 375-376] (Proposed by K.R.S. Sastry). Prove that the nine-point circle of a triangle trisects a median if and only if the side lengths of the triangle are proportional to the lengths of its medians in some order.

This completes the list of results on root-mean-square triangles that I found in the problems pages of **CRUX with MAYHEM**. Triangles for which $2b = c + a$ and triangles for which $2B = C + A$ have likewise been enthusiastically investigated in this journal. We will discuss them in future columns.

References

[1] Leon Bankoff, Solution to Problem 91 (A Long Probate), *Mathematics Magazine*, **41**:5 (Nov-Dec 1968) 291-293.

[2] Leonard Eugene Dickson, *History of the Theory of Numbers*, vol. II, Chelsea, 1952.

[3] Roger A. Johnson, *Modern Geometry*, Houghton Mifflin, 1929; reissued as *Advanced Euclidean Geometry*, Dover, 1960.

[4] *Mathesis*: (1902) 205-208; (1903) 196-200, 226-230, 245-248; (1926) 68-69.

[5] K.R.S. Sastry, Self-Median Triangles, *Mathematical Spectrum*, **22** (1989/90) 58-60.

[6] K.R.S. Sastry, Problem 473, *College Mathematics Journal*, **23**:2 (March 1992) 162; **24**:2 (March 1993) 186-188.