THE OLYMPIAD CORNER

No. 295

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The problems from this issue come from the Italian Team Selection Test, the British Mathematical Olympiad, the Macedonian Mathematical Olympiad, the China Western Mathematical Olympiad, the Austrian Mathematical Olympiad, the Olimpiadi Italiane della Matematica and the Chinese Mathematical Olympiad. Our thanks go to Adrian Tang for sharing the material with the editor.

The solutions to the problems are due to the editor by **1 August 2012**.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

OC21. A sequence of real numbers $\{a_n\}$ is defined by $a_0 \neq 0, 1$, $a_1 = 1 - a_0$, and $a_{n+1} = 1 - a_n(1 - a_n)$ for $n = 1, 2, \cdots$. Prove that for any positive integer n, we have

$$a_0 a_1 \cdots a_n \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) = 1 \,.$$

OC22. Consider a standard 8×8 chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

OC23. Determine all nonnegative integers n such that

$$n(n-20)(n-40)(n-60)\cdots r + 2009$$

is a perfect square where r is the remainder when n is divided by 20.

OC24. Let O be the circumcentre of the triangle ABC. Let K and L be the intersection points of the circumcircles of the triangles BOC and AOC with the bisectors of the angles at A and B respectively. Let P be the midpoint of KL, M symmetrical to O relative to P and N symmetrical to O relative to KL. Prove that $KLMN$ is cyclic.

OC25. Show that the inequality $3^{n^2} > (n!)^4$ holds for all positive integers n.

OC26. Find all functions f from the real numbers to the real numbers which satisfy

$$f\left(x^3\right) + f\left(y^3\right) = (x+y)(f\left(x^2\right) + f\left(y^2\right) - f(xy))$$

for all real numbers x and y.

OC27. A natural number k is said to be *n-squared* if, for every colouring of the squares in a chessboard of size $2n \times k$ with n colours, there are 4 squares with the same colour whose centres are the vertices of a rectangle with sides parallel to the sides of the chessboard.

For any given n, find the smallest natural number k which is n-squared.

OC28. A flea is initially at the point $(0,0)$ of the Euclidean plane. It then takes n jumps. Each jump is taken in any of the four cardinal directions (north, east, south or west). The first jump has length 1, the second jump has length 2, the third jump has length 4, and so on, the n^{th} jump has length 2^{n-1}.

Prove that if we know the number of jumps and the final position, we can uniquely determine the path the flea took.

OC29. Let $n \geq 3$ be a given integer, and $a_1, a_2, \cdots, a_n$ be real numbers satisfying $\min\limits_{1 \leq i < j \leq n} |a_i - a_j| = 1$. Find the minimum value of $\sum\limits_{k=1}^{n} |a_k|^3$.

OC30. Let P be an interior point of a regular n-gon $A_1A_2\cdots A_n$. The lines A_iP meet $A_1A_2\cdots A_n$ at another point B_i, where $i = 1, 2, \cdots, n$. Prove that

$$\sum_{i=1}^{n} PA_i \geq \sum_{i=1}^{n} PB_i \,.$$

. . . .

OC21. On définit une suite de nombres réels $\{a_n\}$ par $a_0 \neq 0, 1$, $a_1 = 1 - a_0$, et $a_{n+1} = 1 - a_n(1 - a_n)$ pour $n = 1, 2, \cdots$. Montrer que pour tout entier positif n, on a

$$a_0 a_1 \cdots a_n \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) = 1 \,.$$

OC22. On considère un échiquier standard 8×8 formé de 64 petits carrés de couleur et disposition usuelles, soit 32 blancs et 32 noirs. Un chemin en *zigzag* sur l'échiquier est une collection de huit carrés blancs, un par rangée, se touchant à leurs coins. Combien de chemins en zigzag y-a-t'il ?

OC23. Trouver tous les entiers non négatifs n tels que

$$n(n-20)(n-40)(n-60)\cdots r+2009$$

soit un carré parfait où r est le reste de la division de n par 20.

OC24. Soit O le centre du cercle circonscrit du triangle ABC. Soit K et L les points d'intersection respectifs des cercles circonscrits BOC et AOC avec les bissectrices des angles en A et B. Soit P le milieu de KL, M le symétrique de O par rapport à P et N le symétrique de O par rapport à KL. Montrer que les points $KLMN$ sont cocycliques.

OC25. Montrer que l'inégalité $3^{n^2} > (n!)^4$ est valable pour tous les entiers positifs n.

OC26. Trouver toutes les fonctions f d'une variable à valeurs réelles et satisfaisant

$$f\left(x^3\right)+f\left(y^3\right)=(x+y)(f\left(x^2\right)+f\left(y^2\right)-f(xy))$$

pour tous les nombres réels x et y.

OC27. Un nombre naturel k est appelé n-*carré* si, pour tout coloriage des cases d'un échiquier de dimension $2n\times k$ avec n couleurs, il y a 4 cases de même couleur dont les centres sont les sommets d'un rectangle dont les côtés sont parallèles à ceux de l'échiquier.

Pour tout n donné, trouver le plus petit nombre naturel k qui soit n-carré.

OC28. On imagine une puce à l'origine $(0,0)$ du plan euclidien. La voilà qui effectue n sauts. Chaque saut a lieu dans l'une quelconque des quatre directions cardinales (nord, est, sud ou ouest). Les longueurs des sauts consécutifs sont, dans l'ordre, de 1, 2, 4 et ainsi de suite, le n-ième saut étant de 2^{n-1}.

Montrer que si l'on connaît le nombre de sauts et la position finale, on peut en déduire univoquement le chemin suivi par la puce.

OC29. On donne un entier $n\geq 3$ et soit $a_1,a_2,\cdots,a_n$ des nombres réels satisfaisant $\min\limits_{1\leq i<j\leq n}|a_i-a_j|=1$. Trouver la valeur minimale de $\sum\limits_{k=1}^{n}|a_k|^3$.

OC30. Soit P un point intérieur d'un polygone régulier $A_1A_2\cdots A_n$ à n côtés. Les droites A_iP coupent ce polygone en un autre point B_i, où $i=1,2,\cdots,n$. Montrer que

$$\sum_{i=1}^{n}PA_i\geq\sum_{i=1}^{n}PB_i\,.$$

Next we turn to our file of solutions from readers to problems of the Youth Mathematical Olympiad of the Asociación Venezolana de Competencias Matemáticas, 2006, given at [2009 : 380].

1. A positive integer has 223 digits and the product of these digits is 3^{446}. What is the sum of the digits?

Solved by Geoffrey A. Kandall, Hamden, CT, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Manes.

Let n be the positive integer with 223 digits. Then the maximum product of the digits of n is $9^{223} = 3^{446}$ when all of the digits of n are nines. Therefore, the digits of n consist of 223 nines and so, the sum of the digits is $9(223) = 2007$.

2. Find all solutions of the equation $m^2 - 3m + 1 = n^2 + n - 1$, where m and n are positive integers.

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Bataille.

The solutions are the pairs $(m, n) = (a+2, a)$ where a is a positive integer. The equation rewrites as $(m-\frac{3}{2})^2 - \frac{5}{4} = (n+\frac{1}{2})^2 - \frac{5}{4}$ or $(m-\frac{3}{2})^2 - (n+\frac{1}{2})^2 = 0$ that is,

$$(m + n - 1)(m - n - 2) \;=\; 0.$$

Thus positive integers m, n are solutions if and only if $m - n - 2 = 0$. The result follows.

3. Define the sequence a_1, a_2, a_3, $\dots$ as follows: let $a_1 = a_2 = 1003$; $a_3 = a_2 - a_1 = 0$; $a_4 = a_3 - a_2 = -1003$; and in general $a_{n+1} = a_n - a_{n-1}$ for any $n \geq 2$. Compute the sum of the first 2006 terms of the sequence.

Solved by Oliver Geupel, Brühl, NRW, Germany; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Geupel.

Let us compute the next items of the sequence:

$$a_5 = a_4 - a_3 \;=\; -1003\,, \qquad a_6 = a_5 - a_4 \;=\; 0\,,$$
$$a_7 = a_6 - a_5 \;=\; 1003\,, \qquad a_8 = a_7 - a_6 \;=\; 1003\,.$$

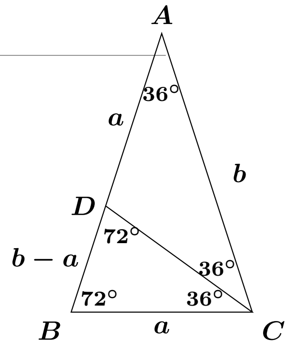
We see that the sequence is periodic with period 6 and $\sum_{n=1}^{6} a_n \;=\; 0$. Noticing that $2006 \equiv 2 \pmod{6}$, we find that the sum of the first 2006 items is

$$\sum_{n=1}^{2006} a_n = a_1 + a_2 = 2006.$$

5. Let ABC be an isosceles triangle with $\angle B = \angle C = 72^\circ$. Find the value of $\dfrac{BC}{AB - BC}$. *Hint:* Consider the bisector CD of $\angle ACB$.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Kandall.

Let CD be the bisector of $\angle ACB$, and let $a = BC$, $b = AB = AC$. We have $\angle BAC = 36^\circ$ and $\angle BDC = 72^\circ$. Therefore, $AD = CD = BC = a$, so $BD = b - a$.



Triangles ABC and CDB are similar, hence $\frac{b}{a} = \frac{a}{b-a} \equiv \lambda$. (Note that $\lambda > 0$.)

We have $\frac{1}{\lambda} = \frac{b-a}{a} = \frac{b}{a} - 1 = \lambda - 1$, hence $\lambda^2 - \lambda - 1 = 0$. Consequently, $\frac{BC}{AB-BC} = \frac{a}{b-a} = \lambda = \frac{1+\sqrt{5}}{2}$.

To finish the material from the October 2009 files we turn to solutions from our readers to problems of the 42$^{\text{nd}}$ Mongolian Mathematical Olympiad, 10$^{\text{th}}$ Grade, given at [2009 : 380–381].

1. Let a, b, c, d, e, and f be positive integers satisfying the relation $ab + ac + bc = de + df + ef$, and let $N = a + b + c + d + e + f$. Prove that if $N \mid (abc + def)$, then N is a composite number.

Solution by Jan Verster, Kwantlen University College, BC.

Expanding the following polynomial, and using the relation $ab + ac + bc = de + df + ef$, we get

$$\begin{aligned}(x + a)(x + b)(x + c) &- (x - d)(x - e)(x - f)\\ &= x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc\\ &\qquad - x^3 + (d + e + f)x^2 - (de + df + ef)x + def\\ &= Nx^2 + abc + def\end{aligned}$$

Then, if we let $x = d$, say, this becomes

$$(d + a)(d + b)(d + c) = Nd^2 + (abc + def)$$

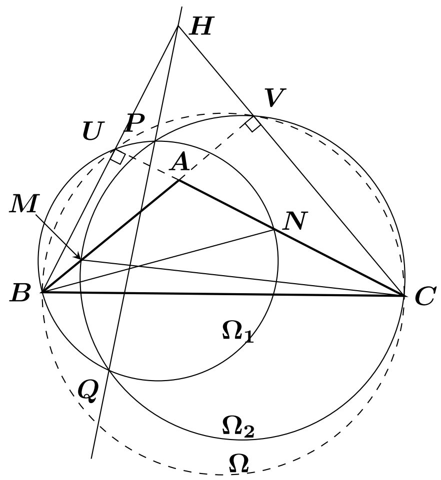
Thus, if $N|(abc + def)$, we also have $N|(d + a)(d + b)(d + c)$. Let p be a prime such that $p|N$. Then p must divide at least one of $d + a$, $d + b$ or $d + c$. Thus $p \le \max(d + a, d + b, d + c) < N$, so p is a proper factor of N, and N must be composite.

Now we turn to solutions from readers to problems of the Olympiade Suisse de mathématiques 2005, tour final, given at [2009 : 82–83].

8. Soint ABC un triangle aigu. Soient M et N des points arbitraires sur les côtés AB et AC respectivement. Les cercles de diamètre BN et CM se coupent en P et Q. Montrer que les points P, Q et l'orthocentre du triangle ABC se trouvent sur une droite.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Michel Bataille, Rouen, France. We give the write-up of Amengual Covas.

Let Ω, Ω_1, and Ω_2 the circles on BC, BN, and CM as diameters, respectively.



Since Ω_1 and Ω_2 intersect at P and Q, the line PQ is the radical axis of Ω_1 and Ω_2.

Let H denote the orthocenter of $\triangle ABC$ and let U and V be the feet of the altitudes from B and C respectively.

Since $\angle BUN = \angle BUC = 90°$, both the circles Ω and Ω_1 pass through U. Hence the line BU is the radical axis of Ω and Ω_1.

Similarly, the line CV is the radical axis of Ω and Ω_2.

Since BU and CV intersect at H, the orthocenter of $\triangle ABC$ is the radical center of Ω, Ω_1, and Ω_2. Hence H lies on the radical axis of Ω_1 and Ω_2, that is, H lies on the line PQ. This completes the proof of the collinearity of P, Q and the orthocenter of $\triangle ABC$.

As shown in the proof, the condition $\triangle ABC$ acute is not necessary.

Next we finish up the solutions to problems of the 55^{th} Czech and Slovak Mathematical Olympiad 2006 given at [2009: 81–82].

6. (J. Švrček, P. Calábek) Solve in real numbers the system of equations

$$\left.\begin{aligned} \tan^2 x + 2\cot^2 2y &= 1\,, \\ \tan^2 y + 2\cot^2 2z &= 1\,, \\ \tan^2 z + 2\cot^2 2x &= 1\,. \end{aligned}\right\} \qquad (1)$$

Solved by Arkady Alt, San Jose, CA, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Zelator's write-up.

First note that the real numbers $x = k\pi \pm \frac{\pi}{4}$, $y = m\pi \pm \frac{\pi}{4}$, $z = n\pi \pm \frac{\pi}{4}$; where k, m, n are (arbitrary) integers; are solutions of the system (1).

We see, by inspection, that any solutions must satisfy $0 \leq \tan^2 x$, $\tan^2 y$, $\tan^2 z \leq 1$ and $0 \leq \cot^2 2y$, $\cot^2 2x$, $\cot^2 2z \leq \frac{1}{2}$. However, in fact none of $\tan x$, $\tan y$, $\tan z$ can be zero, since if $\tan x = 0$; then, and only then $x = t\pi$, for some $t \in \mathbb{Z}$. But then $2x = 2t\pi$; and so $\cot 2x$ would be undefined. Thus,

$$\left.\begin{array}{rcl} 0 & < & \tan^2 x\,, \quad \tan^2 y\,, \quad \tan^2 z \ \leq \ 1 \\ \text{and } 0 & \leq & \cot^2 2y, \cot^2 2x, \cot^2 2z \ < \ \frac{1}{2} \end{array}\right\} \tag{2}$$

Next, observe if one of $\tan^2 x$, $\tan^2 y$, $\tan^2 z$ is equal to 1; then all three of them are and the three cotangent terms are zero. Indeed, suppose that $\tan^2 x = 1$; then $\tan x = 1$ or -1; which implies $x = k\pi \pm \frac{\pi}{4}$; $k \in \mathbb{Z}$. But then from the first equation in (1) we obtain $\cot 2y = 0$; $2y = \rho\pi + \frac{\pi}{2}$, $\rho \in \mathbb{Z}$; $y = \frac{\rho}{2}\pi + \frac{\pi}{4}$; when $\rho =$ even $= 2\lambda$; $y = \lambda\pi + \frac{\pi}{4}$; and when $\rho =$ odd $= 2\lambda + 1$; $y = \lambda\pi + \frac{3\pi}{4}$. These two formulas, since $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$; can be condensed into one: $y = m\pi \pm \frac{\pi}{4}$; $m \in \mathbb{Z}$.

Then $\tan^2 y = 1$; and from the second equation in (1), we obtain $\cot 2z = 0$; and from which (via a similar argument) we obtain $z = n\pi \pm \frac{\pi}{4}$; $n \in \mathbb{Z}$. We conclude that either $\tan^2 x = \tan^2 y = \tan^2 z = 1$, which produces the solutions $x = k\pi \pm \frac{\pi}{4}$, $y = m\pi \pm \frac{\pi}{4}$, $z = n\pi \pm \frac{\pi}{4}$. Or alternatively,

$$\left.\begin{array}{rcl} 0 & < & \tan^2 x\,, \quad \tan^2 y\,, \quad \tan^2 z \ < \ 1 \\ \text{and } 0 & < & \cot^2 2y\,, \quad \cot^2 2x\,, \quad \cot^2 2z \ < \ \frac{1}{2} \end{array}\right\} \tag{3}$$

Below, we shall find all the real solutions to system (1) that satisfy the conditions in (3). Suppose (x_1, y_1, z_1) is a solution satisfying (3). We put $r_1 = \tan^2 x_1$, $r_2 = \tan^2 y_1$, and $r_3 = \tan^2 z_1$. So that by (3),

$$\left.\begin{array}{rcl} 0 & < & r_1 \ = \ \tan^2 x_1\,, \quad r_2 \ = \ \tan^2 x_1\,, \quad r_3 \ = \ \tan^2 z_1 \ < \ 1 \\ \text{and } 0 & < & \cot^2 2y_1\,, \quad \cot^2 2x_1\,, \quad \cot^2 2z_1 \ < \ \frac{1}{2} \end{array}\right\} \tag{4}$$

Since (x_1, y_1, z_1) satisfies the system (1) we have,

$$\left.\begin{array}{rcl} \tan^2 x_1 + 2\cot^2 2y_1 & = & 1 \\ \tan^2 y_1 + 2\cot^2 2z_1 & = & 1 \\ \tan^2 z_1 + 2\cot^2 2x_1 & = & 1 \end{array}\right\} \tag{5}$$

Consider the first equation in (12) and multiply across by $\tan^2 2y_1$ in order to obtain, since $\cot^2 2y_1 \cdot \tan^2 2y_1 = 1$,

$$\tan^2 2y_1 \cdot \tan^2 x_1 + 2 \ = \ \tan^2 2y_1\,.$$

Applying the double-angle identity $\tan 2y_1 = \frac{2\tan y_1}{1-\tan^2 y_1}$ gives $4\tan^2 y_1 \tan^2 x_1 + 2(1 - \tan^2 y_1)^2 \ = \ 4\tan^2 y_1$.

$$\Leftrightarrow \tan^4 y_1 + 2\tan^2 y_1 \tan^2 x_1 - 4\tan^2 y_1 + 1 \ = \ 0;$$

(with $r_1 = \tan^2 x_1$, $r_2 = \tan^2 y_1$) $r_2^2 + 2r_2r_1 - 4r_2 + 1 = 0$. Working similarly with the second and third equations in (12) we altogether obtain

$$\left.\begin{aligned} r_2^2 + 2r_2r_1 - 4r_2 + 1 &= 0 \\ r_3^2 + 2r_3r_2 - 4r_3 + 1 &= 0 \\ r_1^2 + 2r_1r_3 - 4r_1 + 1 &= 0 \end{aligned}\right\} \tag{6}$$

Adding the three equations in (6) yields

$$(r_1 + r_2 + r_3)^2 - 4(r_1 + r_2 + r_3) + 3 = 0;$$

or equivalently,

$$[(r_1 + r_2 + r_3) - 1][(r_1 + r_2 + r_3) - 3] = 0\,. \tag{7}$$

By (4), it follows that $0 < r_1 + r_2 + r_3 < 3$; which implies in conjunction with (7) that

$$r_1 + r_2 + r_3 = 1 \tag{8}$$

Now, we go to (6) and we multiply the first equation by r_1r_3, the second equation by r_1r_2; and the third equation by r_2r_3, in order to obtain

$$\left.\begin{aligned} r_1r_3r_2^2 + 2r_2r_3r_1^2 - 4r_1r_2r_3 + r_1r_3 &= 0 \\ r_1r_2r_3^2 + 2r_1r_3r_2^2 - 4r_1r_2r_3 + 4_14_2 &= 0 \\ r_2r_3r_1^2 + 2r_1r_2r_3^2 - 4r_1r_2r_3 + r_2r_3 &= 0 \end{aligned}\right\} \tag{9}$$

We add the three equations in (9) to get,

$$\begin{aligned} &r_1r_2r_3(r_1 + r_2 + r_3) + 2r_1r_2r_3(r_1 + r_2 + r_3) \\ &\quad -12r_1r_2r_3 + (r_1r_3 + r_1r_2 + r_2r_3) = 0; \end{aligned}$$

and by (8) we arrive at

$$r_1r_3 + r_1r_2 + r_2r_3 = 9r_1r_2r_3 = 9P \text{ (product).} \tag{10}$$

By the Arithmetic-Geometric Mean inequality we also have,

$$r_1r_3 + r_1r_2 + r_2r_3 \geq 3\sqrt[3]{(r_1r_3)(r_1r_2)(r_2r_3)};$$

and by (10);

$$9P \geq 3P^{2/3} \Leftrightarrow P^{1/3} \geq \frac{1}{3};$$

or equivalently,

$$P \geq \frac{1}{27}\,. \tag{11}$$

Next, consider the cubic polynomial of degree three,

$$f(t) \;=\; (t-r_1)(t-r_2)(t-r_3); \tag{12}$$

$$f(t) = t^3 - (r_1+r_2+r_3)t^2 + (r_1r_2+r_2r_3+r_3r_1)t - r_1r_2r_3;$$

and by (8) and (10);

$$f(t) \;=\; t^3 - t^2 + 9P\cdot t - P\,. \tag{13}$$

Next, we make use of the following lemma. The facts stated in the Lemma are well-known, standard material on cubic polynomial functions.

Lemma 1. Suppose that $g(t)$ is a cubic polynomial function of degree 3 with leading coefficient $a > 0$; and let $g'(t)$ be the derivative of $g(t)$; $g'(t)$ being a quadratic trinomial.

(i) If the discriminant of $g'(t)$ is negative; then the function $g(t)$ has no critical numbers in its domain, and therefore it has no points of local maximum or local minimum. The function $g(t)$ is increasing throughout $\mathbb{R}$, and has only one inflection point. And the function $g(t)$ has *exactly one real root*, and two conjugate complex roots.

(ii) If the discriminant of $g'(t)$ is zero; then the function $g(t)$ has exactly one critical number ρ in its domain $\mathbb{R}$. The point $(\rho, g(\rho))$ is both an inflection point and a critical point on the graph of $g(t)$. The function $g(t)$ increases throughout $\mathbb{R}$, it has no points of local maximum or minimum. Furthermore $g(t)$ has the form, $g(t) = a(t-\rho)^3 + \kappa$, for some $\kappa \in \mathbb{R}$. Thus, $g(t)$ has a triple real root, the real number

$$r \;=\; \sqrt[3]{-\frac{\kappa}{a}} + \rho\,.$$

Consider the derivative of $f(t)$ in (13): $f'(t) = 3t^2 - 2t + 9P$. The discriminant of $f'(t)$ is $D = 4 - 4\cdot 3\cdot 9\cdot P = 4(1-27P)$. By (11) it follows that $D = 0$ or $D < 0$. The second possibility, $D < 0$ is eliminated by Lemma 1(i), since according to (12), $f(t)$ has three real roots. Thus we must have $D = 0$.

By Lemma 1(ii), it follows that $r_1 = r_2 = r_3$; and therefore from (8) we obtain $r_1 = r_2 = r_3 = \frac{1}{3}$. Back to (4):

$$\tan^2 x_1 = \frac{1}{3} \;\Leftrightarrow\; \tan x_1 \;=\; \pm\frac{1}{\sqrt{3}}; \qquad x_1 \;=\; k\pi \pm \frac{\pi}{6}; \qquad \text{for some } k \in \mathbb{Z}\,.$$

Likewise, we obtain

$$y_1 \;=\; m\pi \pm \frac{\pi}{6}\,, \qquad \text{for } m \in \mathbb{Z}\,,$$

and

$$z_1 \;=\; n\pi \pm \frac{\pi}{6}\,, \qquad \text{for } n \in \mathbb{Z}\,.$$

Conversely, one can verify directly that if the triple (x_1, y_1, z_1) has the above form; then it satisfies the system (1).

Conclusion. The set of solutions S of system (1) is the union of two disjoint families or solution sets S_1, S_2;

$$S = S_1 \cup S_2 ,$$

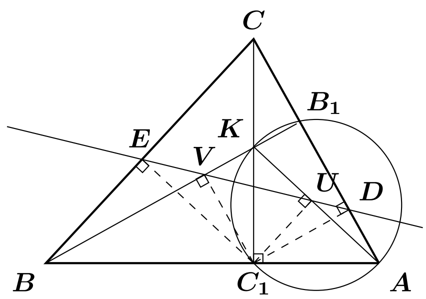
where $S_1 = \{(x, y, z) \mid x, y, z \in \mathbb{R}$ and such that $x = k\pi \pm \frac{\pi}{4}$, $y = m\pi \pm \frac{\pi}{4}$, $z = n\pi \pm \frac{\pi}{4}\}$; where m, n, k can be any integers; and all eight combinations of signs are allowed. And $S_2 = \{(x, y, z) \mid x, y, z \in \mathbb{R}$ and such that $x = k\pi \pm \frac{\pi}{6}$, $y = m\pi \pm \frac{\pi}{6}$, $z = n\pi \pm \frac{\pi}{6}\}$; where m, n, k can be any integers; and all eight combinations of signs are allowed.

7. (I. Voronovich) The point K (distinct from the orthocentre) lies on the altitude CC_1 of the acute triangle ABC. Prove that the feet of the perpendiculars from C_1 to the segments AC, BC, BK, and AK lie on a circle.

Solution by Michel Bataille, Rouen, France.

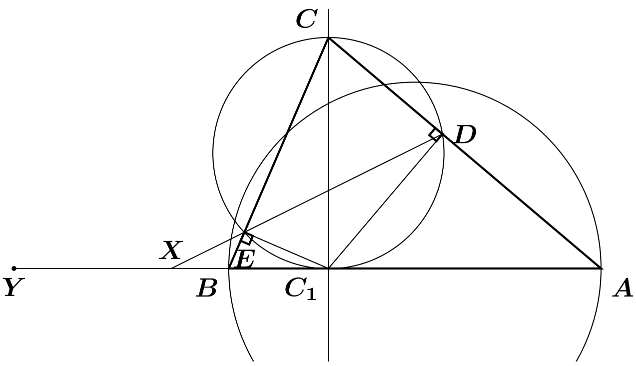
We denote by D, U, V, E the feet of the perpendiculars from C_1 to AC, AK, BK, BC, respectively, and by H the orthocentre of ΔABC.

First, we show that D, U, V, E are not collinear. Assuming the contrary, let BK meet AC at B_1. From Simson's theorem, C_1 lies on the circumcircle of ΔAKB_1. Since $\angle AC_1K = 90°$, it follows that AK is a diameter of the circle (AKB_1) and so $\angle KB_1A = 90°$. Thus BB_1 is the altitude from B of ΔABC and K is its orthocentre, contradicting the hypothesis.



If $CA = CB$, since the line CC_1 is an axis of symmetry of the figure, $DUVE$ is an isosceles trapezium and D, U, V, E are concyclic. From now on, we assume that $CA \neq CB$.

Let X be the point of intersection of the lines DE and AB. Since the circle with diameter CC_1 passes through D and E, we have $XD \cdot XE = XC_1^2$. On the other hand, since $\angle XEB = \angle CED = \angle CC_1D = \angle DAX$, the triangle XBE and XDA are similar.



It follows that $XD \cdot XE = XA \cdot XB$ and so

$$XA \cdot XB \;=\; XC_1^2. \tag{1}$$

Now, if Y is the pole of the line CC_1 with respect to the circle with diameter AB, points Y, C_1 are harmonic conjugates with respect to A, B and so (1) characterizes X as being the midpoint of YC_1. Reasoning in the same way with triangle KAB instead of CAB, we see that the line UV passes through X as well and that $XU \cdot XV = XC_1^2 = XD \cdot XE.$ Since U, V, D, E are not collinear, the relation $XU \cdot XV = XD \cdot XE$ implies that U, V, D, E are concyclic, as required.

8. (E. Barabanov, V. Kaskevich, S. Mazanik, I. Voronovich) An equilateral triangle of side n is divided into n^2 unit equilateral triangles by lines parallel to its sides. Determine the smallest possible number of small triangles that must be marked so that any unmarked triangle has at least one side in common with a marked triangle.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let T_n be the smallest number of marked triangles. We prove: $T_n = \left\lceil \frac{n^2}{4} \right\rceil$.

Since each marked triangle has at most three neighboring unmarked triangles, we have $3T_n \geq n^2 - T_n$; hence $T_n \geq \left\lceil \frac{n^2}{4} \right\rceil$. Vice versa, we show that

$$T_n \;\leq\; \left\lceil \frac{n^2}{4} \right\rceil. \tag{1}$$

The proof is by induction. The relation (1) holds for $n \in \{1, 2\}$; see Figure 1 for $n = 2$.
Assume that(1) holds for each $n \in \{1, 2, \ldots, N-1\}$.
Let Δ_k denote an equilateral triangle of side length k.
We make three cases.

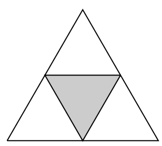


Figure 1

Case 1: N is even.

The triangle Δ_N can be covered by one Δ_{N-2} and $N-1$ triangles Δ_2 (Figure 2). By induction,

$$T_N \;\leq\; T_{N-2} + (N-1)T_2 \;\leq\; \left(\frac{N}{2} - 1\right)^2 + N - 1 \;=\; \left\lceil \frac{N^2}{4} \right\rceil.$$

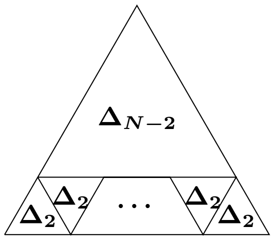


Figure 2

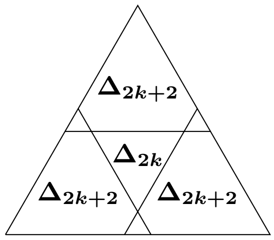


Figure 3

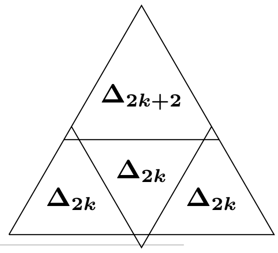


Figure 4

Case 2: $N = 4k + 3 \quad (k \geq 0)$.

The triangle Δ_N can be covered by one Δ_{2k} and three Δ_{2k+2} (Figure 3). By induction,

$$T_N \;\leq\; T_{2k} + 3T_{2k+2} \;\leq\; k^2 + 3(k+1)^2 \;=\; \left\lceil \frac{(4k+3)^2}{4} \right\rceil .$$

Case 3: $N = 4k + 1 \quad (k \geq 1)$. The triangle Δ_N can be covered by three Δ_{2k} and one Δ_{2k+2} (Figure 4). By induction,

$$T_N \;\leq\; 3T_{2k} + T_{2k+2} \;\leq\; 3k^2 + (k+1)^2 \;=\; \left\lceil \frac{(4k+1)^2}{4} \right\rceil .$$

This completes the proof.

Next we look at readers' solutions to problems given in the November 2009 number of the *Corner* and the 24th Iranian Mathematical Olympiad, First Round, given at [2009 : 435].

1. Given integers $m > 2$ and $n > 2$, prove there is a sequence of integers a_0, a_1, $\ldots$, a_k such that $a_0 = m$, $a_k = n$, and $(a_i + a_{i+1}) \mid (a_i a_{i+1} + 1)$ for each $i = 0, 1, \ldots, k-1$.

Solution by Titu Zvonaru, Comăneşti, Romania.

Taking $a_0 = m$, $a_1 = 1$, $a_2 = 1, \ldots, a_{k-1} = 1$, $a_k = n$ we have

$$\begin{aligned} a_0 + a_1 \;&=\; m+1; \quad a_0 a_1 + 1 \;=\; m+1 \\ a_1 + a_2 \;&=\; 2; \quad a_1 a_2 + 1 \;=\; 2 \\ &\;\vdots \quad \vdots \\ a_{k-2} + a_{k-1} \;&=\; 2; \quad a_{k-2} a_{k-1} + 1 \;=\; 2 \\ a_{k-1} + a_k \;&=\; n+1; \quad a_{k-1} a_k + 1 \;=\; n+1 \end{aligned}$$

hence $(a_i + a_{i+1}) \mid (a_i a_{i+1} + 1)$ for each $i = 0, 1, \ldots, k-1$.

4. Find all two-variable polynomials $p(x, y)$ with real coefficients such that $p(x + y, x - y) = 2p(x, y)$ for all real numbers x and y.

Solution by Arkady Alt, San Jose, CA, USA.

Note that $p(2x, 2y) = p((x + y) + (x - y), (x + y) - (x - y)) = 2p(x + y, x - y) = 4p(x, y)$. Excluding the trivial case $p(x, y) \equiv 0$ we assume further that $p(x, y) \neq 0$.

Since $p(0, 0) = p(2 \cdot 0, 2 \cdot 0) = 4p(0, 0)$ then $p(0, 0) = 0$ and $p(x, y)$ is not constant, moreover $p(x, 0)$ and $p(0, y)$ are not constants. Note that any such two-variable polynomial $p(x, y)$ can be represented in the form $p(x, y) = A(x) + B(y) + xyC(x, y)$, where $\deg A(x) = n > 0, \deg B(y) = m > 0,$ more precisely $A(x) = p(x, 0) = a_n x^n + a_{n-1}x^{n-2} + \ldots + a_1 x,\ B(y) = p(0, y) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x$, where $a_n \neq 0, b_m \neq 0.$

Since $p(2x, 2y) = 4p(x, y)$ then in particular for $y = 0$ and any real x we have $p(2x, 0) = 4p(x, 0)$ if and only if $2^n a_n = 4a_n, 2^{n-1}a_{n-1} = 4a_{n-1}, \ldots, 2a_1 = 4a_1$ if and only if $n = 2, k_1 = 0,$ Similarly we obtain $m = 2, b_1 = 0.$

Thus, $p(x, y) = ax^2 + xyC(x, y) + by^2.$ Since $p(x, x) = x^2(a + b + C(x, x))$ and $p(2x, 2x) = 4p(x, x)$ then for $x \neq 0$ we have $4x^2(a + b + C(2x, 2x)) = 4x^2(a + b + C(x, x))$ if and only if $C(2x, 2x) = C(x, x)$ if and only if $C(x, x)$ is constant.

Indeed, since $C(x, x) = c + c_1x^2 + \cdots + c_k x^{2k}$ then $C(2x, 2x) = C(x, x)$ if and only if $c_i = 2^{2i}c_i, i = 1, 2, \ldots, k$ if and only if $c_i = 0,\ i = 1, 2, \ldots, k.$ So, $p(x, y) = ax^2 + cxy + by^2$ and since $p(x + y, x - y) = 2p(x, y)$ if and only if $a(x + y)^2 + c(x^2 - y^2) + b(x - y)^2 = 2ax^2 + 2cxy + 2by^2$ if and only if $(b + c - a)x^2 + (a - b - c)y^2 + 2(a - c - b)xy = 0$ for any x, y then $c = a - b.$

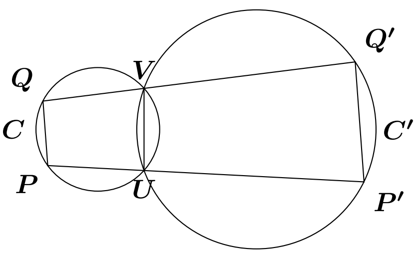
Therefore, $p(x, y) = ax^2 + (a - b)xy + by^2$ and all such two-variable polynomials $p(x, y)$ of the second degree satisfy $p(x + y, x - y) = 2p(x, y).$

Indeed, $p(x + y, x - y) = ax^2 + 2axy + ay^2 + bx^2 - 2bxy + by^2 + (b - a)(x^2 - y^2) = 2(ax^2 + (a - b)xy + by^2) = 2p(x, y).$

5. Let ω_1 and ω_2 be two circles such that the centre of ω_1 is located on ω_2. If the circles intersect at M and N, AB is an arbitrary diameter of ω_1, and A_1 and B_1 are the second intersections of AM and BN with the circle ω_2 (respectively), prove that A_1B_1 is equal to the radius of ω_1.

Solution by Michel Bataille, Rouen, France.

The following lemma will be used twice: Let C and C' be two circles intersecting at U, V. If points P, Q on C and P', Q' on C' are such that P, U, P' and Q, V, Q' are collinear, then PQ and $P'Q'$ are parallel.

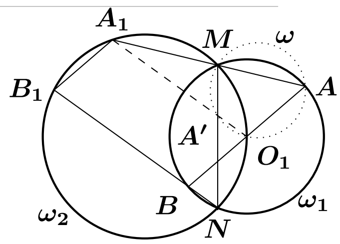


Indeed, denoting by $\angle(\cdot,\cdot)$ the directed angle of lines, we have (modulo π)

$$\begin{aligned}\angle(PQ, P'Q') &= \angle(PQ, PU) + \angle(P'U, P'Q') \\ &= \angle(VQ, VU) + \angle(VU, VQ') = \angle(VQ, VQ') = 0\end{aligned}$$

where the second equality follows from the concyclicity of P, Q, U, V and P', Q' ,U ,V.

This lemma clearly implies that A_1B_1 and AB are parallel. Now, consider the inversion $\mathbf{I}$ in the circle ω_1. Since $\mathbf{I}(M) = M$ and $\mathbf{I}(N) = N$, $\mathbf{I}$ exchanges the line MN and the circle ω_2. It follows that $\mathbf{I}(A_1) = A'$ is the point of intersection of MN and O_1A_1.



We remark that A' is also on the the circumcircle ω of ΔO_1AM, which is the image of AM under $\mathbf{I}$. Applying again the lemma, but this time to the circles ω and ω_1, we obtain $BN \parallel O_1A'$. As a result, we have $A_1B_1 \parallel O_1B$ and $BB_1 \parallel O_1A_1$ so that $A_1B_1BO_1$ is a parallelogram and $A_1B_1 = O_1B$, the radius of ω_1.

Next we look at solutions from the file for the 24$^{\text{th}}$ Iranian Mathematical Olympiad, Third Round, given at [2009 : 437–438].

2. Does there exist a sequence of positive integers a_0, a_1, a_2, $\dots$ such that $\gcd(a_i, a_j) = 1$ whenever $i \neq j$, and for all n the polynomial $\sum\limits_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$?

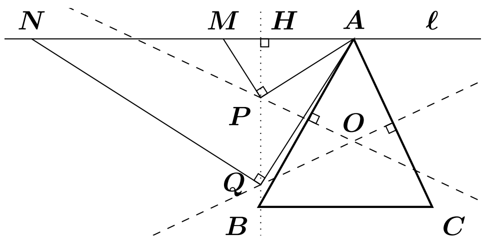
Solved by Mohammed Aassila, Strasbourg, France.

Consider, for example, $a_n = p_n$ where (p_n) is a sequence of prime numbers such that $p_n > p_0 + p_1 + \cdots + p_{n-1}$. Then, by this condition, all the roots of $\sum_{i=0}^{n} p_i z^i$ lie inside the unit circle $|z| < 1$.

3. Triangle ABC is isosceles with $AB = AC$. The line ℓ passes through A and is parallel to BC. The points P and Q are on the perpendicular bisectors of AB and AC, respectively, and such that $PQ \perp BC$. The points M and N are on ℓ and such that $\angle APM$ and $\angle AQN$ are right angles. Prove that

$$\frac{1}{AM} + \frac{1}{AN} \le \frac{2}{AB}\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give the solution by Bataille.



Let H be the orthogonal projection of P (or Q) onto ℓ. Since triangles ΔAPM and ΔAQN are right-angled at P and Q, respectively, we have $AP^2 = AH \cdot AM$ and $AQ^2 = AH \cdot AN$, hence

$$\frac{1}{AM} = \frac{AH}{AP^2} \text{ and } \frac{1}{AN} = \frac{AH}{AQ^2}.$$

Now, let O denote the circumcentre of ΔABC. Then, from $\angle OPQ = \angle ABC = B$ and $\angle PQO = \angle ACB = C = B$ (acute angles with perpendicular sides) we deduce $OP = OQ$ and $\angle POA = B, \angle AOQ = 180^\circ - B$. Using the law of cosines and denoting by R the circumradius of ΔABC, it follows that

$$\begin{aligned} AP^2 &= OP^2 + R^2 - 2R \cdot OP \cdot \cos B \\ &= (OP - R)^2 + 2R \cdot OP(1 - \cos B) \\ &= (OP - R)^2 + 4R \cdot OP \sin^2(B/2) \\ AQ^2 &= OP^2 + R^2 + 2R \cdot OP \cdot \cos B \\ &= (OP - R)^2 + 2R \cdot OP(1 + \cos B) \\ &= (OP - R)^2 + 4R \cdot OP \cos^2(B/2) \end{aligned}$$

and so

$$AP^2 \geq 4R \cdot OP \sin^2(B/2), \qquad AQ^2 \geq 4R \cdot OP \cos^2(B/2).$$

Observing that $\dfrac{AH}{OP} = \sin\angle(POA) = \sin B$, we obtain

$$\begin{aligned} \frac{1}{AM} + \frac{1}{AN} &= \frac{AH}{AP^2} + \frac{AH}{AQ^2} \leq \frac{1}{2R}\left(\frac{\cos(B/2)}{\sin(B/2)} + \frac{\sin(B/2)}{\cos(B/2)}\right) \\ &= \frac{1}{2R\sin(B/2)\cos(B/2)} = \frac{2}{AB}. \end{aligned}$$

(since $AB = 2R\sin B$ by the law of sines). Clearly equality holds if and only if $OP = OQ = R$.

6. Find all polynomials $p(x)$ of degree 3 such that for all nonnegative real numbers x and y

$$p(x+y) \;\geq\; p(x) + p(y)\,.$$

Solved by Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the solution of Bataille.

Let us say that $p(x)$ is superadditive if $p(x+y) \geq p(x) + p(y)$ for all nonnegative x and y. We show that the superadditive polynomials of degree 3 are the polynomials

$$ax^3 + bx^2 + cx + d$$

where $c \in \mathbb{R},\ a > 0,\ d \leq 0$ and $8b^3 \geq 243da^2$.

Let $p(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are real numbers and $a \neq 0$ and let

$$\delta(x, y) = p(x + y) - p(x) - p(y) = 3axy(x + y) + 2bxy - d.$$

Clearly, $p(x)$ is superadditive if and only if $\delta(x, y) \geq 0$ for all $x, y \geq 0$. First, assume that it is the case. Then, $\delta(0, 0) \geq 0$, hence $d \leq 0$. Also, for $x > 0$,

$$3a + \frac{b}{x} - \frac{d}{2x^3} = \frac{\delta(x, x)}{2x^3} \geq 0$$

so that $3a \geq 0$ (by letting x approach infinity) and $a > 0$ is obtained. Lastly, $8b^3 \geq 243da^2$ holds if $b \geq 0$ (since $d \leq 0$) and if $b < 0$ results from

$$0 \leq \delta\left(\frac{-2b}{9a}, \frac{-2b}{9a}\right) = \frac{8b^3 - 243da^2}{243a^2}.$$

Conversely, assume that the conditions $a > 0,\ d \leq 0$ and $8b^3 \geq 243da^2$ hold. Clearly, $\delta(x, y) \geq 0$ for all $x, y \geq 0$ if $b \geq 0$. Suppose that $b < 0$. Observing that

$$\delta(x, y) \geq 3axy \cdot 2\sqrt{xy} + 2bxy - d = \phi(\sqrt{xy})$$

where $\phi(t) = 6at^3 + 2bt^2 - d$, it suffices to show that $\phi(t) \geq 0$ for $t \geq 0$. Since the derivative of ϕ is given by $\phi'(t) = 18at\left(t + \frac{2b}{9a}\right)$, it is readily seen that the minimum of ϕ on $[0, \infty)$ is $\phi\left(\frac{-2b}{9a}\right) = \frac{8b^3 - 243da^2}{243a^2}$, which is nonnegative by assumption. Thus, $\phi(t) \geq 0$ for $t \geq 0$ and the proof is complete.

We continue with solutions from our readers to problems given in the May 2010 number of the *Corner* with the XVIII Olimpiada de Matemática de Paises del Cono Sur given at [2010: 217–218].

1. Find all the pairs (x, y) of nonnegative integers that satisfy

$$x^3y + x + y \ = \ xy + 2xy^2\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Geoffrey A. Kandall, Hamden, CT, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Bataille's write-up.

The solutions are the pairs $(0, 0),\ (1, 1),\ (2, 2)$.

It is readily checked that these pairs are solutions. Conversely, let (x, y) be any solution. Then we have

$$y = x(y + 2y^2 - 1 - x^2y) \tag{1}$$

and

$$x = y(x - x^3 - 1 + 2xy). \qquad (2)$$

From (1), x divides y (since $y + 2y^2 - 1 - x^2y$ is an integer) and from (2), y divides x (since $x - x^3 - 1 + 2xy$ is an integer). It follows that $|x| = |y|$ and since x, y are nonnegative integers, $x = y$.

Now, from the equation, we must have $x^4 + 2x = x^2 + 2x^3$ that is,

$$x(x^2 - 1)(x - 2) = 0,$$

which implies $x \in \{0, 1, 2\}$. The result follows.

2. Given are 100 positive whole numbers whose sum equals their product. Determine the minimum number of occurrences of the number 1 among the 100 numbers.

Solved by Henry Ricardo, Tappan, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the version by Zvonaru.

Let $x_{100} \le x_{99} \le \cdots \le x_2 \le x_1$ be the given 100 positive integers and let $k_{\min}$ be the searched minimum number of 1's.

The equation $x_{100} + x_{99} + \cdots + x_2 + x_1 = x_{100}x_{99} \ldots x_2x_1$ can be rewritten as

$$\frac{x_{100}}{x_{100} \cdot x_{99} \cdots x_2x_1} + \cdots + \frac{x_2}{x_{100} \cdot x_{99} \cdots x_2x_1} + \frac{x_1}{x_{100} \cdot x_{99} \cdots x_2x_1} = 1.$$

Since $x_{100} \le x_{99} \le \cdots \le x_2 \le x_1$, it results that $\frac{x_1}{x_{100} \cdot x_{99} \cdots x_2x_1} \ge \frac{1}{100}$, that is $x_{100} \cdot x_{99} \cdots x_2 \le 100$.

We deduce that at most 6 numbers among $x_{100}, x_{99}, \cdots, x_2$ may be greater than 1, hence $k_{\min} \ge 93$.

Suppose that $k_{\min} = 93$. We must solve the equation

$$93 + x_7 + x_6 + x_5 + x_4 + x_3 + x_2 + x_1 = x_1x_2x_3x_4x_5x_6x_7,$$

where $x_1 \ge x_2 \ge x_3 \ge x_4 \ge x_5 \ge x_6 \ge x_7 \ge 2$.

We have

$$93 + 7x_1 \ge 93 + x_1 + x_2 + x_3 + x_4 + x_5 + x_5 + x_7 = x_1x_2x_3x_4x_5x_6x_7 \ge 2^6x_1$$

that is $57x_1 \le 93$, a contradiction with $x_1 \ge 2$.

Suppose now that $k_{\min} = 94$. As above, we must solve the equation

$$94 + 6x_1 \ge 94 + x_6 + x_5 + x_4 + x_3 + x_2 + x_1 = x_1x_2x_3x_4x_5x_6.$$

We have

$$94 + 6x_1 \ge x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = x_1x_2x_3x_4x_5x_6 \ge 32x_1,$$

that is $26x_1 \le 94$.

If $x_1 = 2$, then $x_2 = x_3 = x_4 = x_5 = x_6$ and we have no solution (because $94 + 6 \cdot 2 \neq 2^6$).

If $x_1 = 3$, we have to solve the equation

$$97 + x_6 + x_5 + x_4 + x_3 + x_2 = 3x_2x_3x_4x_5x_6.$$

We deduce that

$$97 + 5x_2 \geq 97 + x_2 + x_3 + x_4 + x_5 + x_6 = 3x_2x_3x_4x_5x_6 \geq 48x_2$$

and it follows that $43x_2 \leq 97$, hence $x_2 = 2$.

We obtain $x_2 = x_3 = x_4 = x_5 = x_6 = 2$ and we have no solution (because $97 + 5 \cdot 2 \neq 3 \cdot 32$).

Since the equation

$$95 + x_1 + x_2 + x_3 + x_4 + x_5 = x_1x_2x_3x_4x_5$$

has solution $x_1 = x_2 = x_3 = 3$, $x_4 = x_5 = 2$

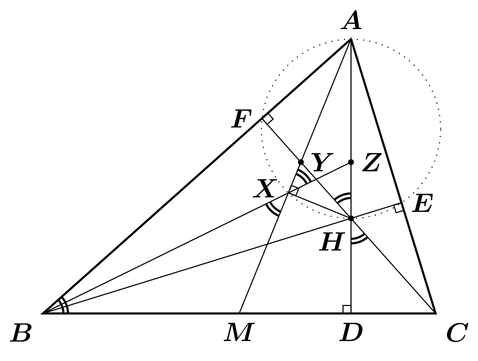
$$(95 + 3 + 3 + 3 + 2 + 2 = 108,\ 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 108),$$

it results that $k_{\min} = 95$.

3. Let ABC be an acute triangle with altitudes AD, BE, CF, where D, E, F lie on BC, AC AB, respectively. Let M be the midpoint of BC. The circumcircle of triangle AEF cuts the line AM at A and X. The line AM cuts the line CF at Y. Let Z be the point of intersection of AD and BX. Show that the lines YZ and BC are parallel.

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Geupel's solution.

Let $\alpha = \angle CAB$, $\beta = \angle ABC$, and let H be the orthocentre of $\triangle ABC$.



Because of $\triangle ABD \sim \triangle AHF$, we have $AD : AB = AF : AH$, so that $AD \cdot AH = AB \cdot AF = AB \cdot AC \cos \alpha$. For the median line AM, we have

$$4AM^2 = 2AB^2 + 2AC^2 - BC^2\,.$$

By the Law of Cosines it now follows that

$$BC^2 = 2AB^2 + 2AC^2 - BC^2 - 4AB \cdot AC \cos \alpha = 4AM^2 - 4AD \cdot AH;$$

hence

$$BM^2 = AM^2 - AD \cdot AH.$$

By $\angle AEH = \angle AFH = 90°$, the point H lies on the circumcircle of quadrilateral $AEXF$. Hence $\angle AXH = \angle AFH = 90°$. Thus $\triangle AHX \sim \triangle AMD$, which implies that $AH : AX = AM : AD$ and therefore

$$\begin{aligned} AM^2 - BM^2 &= AD \cdot AH = AM \cdot AX \\ &= AM(AM - MX) = AM^2 - AM \cdot MX\,. \end{aligned}$$

We obtain $BM^2 = AM \cdot MX$ and therefore $AM : BM = BM : XM$. Noticing that $\angle AMB = \angle BMX$, we obtain $\triangle ABM \sim \triangle BXM$. Hence, $\angle YXZ = \angle BXM = \angle ABM = \beta$. Also, $\angle YHZ = \angle CHD = \angle DBA = \beta$. We obtain $\angle YXZ = \angle YHZ$, so the quadrilateral $HXYZ$ is cyclic, which establishes $\angle HZY = 180° - \angle HXY = 90°$ and $YZ \parallel BC$.

4. Some cells of a 2007×2007 table are coloured. The table is "charrua" if none of the rows and none of the columns are completely coloured.

(a) What is the maximum number k of coloured cells that a charrua table can have?

(b) For such k, calculate the number of distinct charrua tables that exist.

Solved by Titu Zvonaru, Cománeşti, Romania.

(a) We will determine the minimum number k' of uncoloured cells that a charrua table can have.

It is easy to see that this minimum of uncoloured cells is 2007 (if $k' < 2007$, then there is at least a row or a column which is completely coloured).

An example of a charrua table with $k' = 2007$: the uncoloured cells are $c(1,1), c(2,2), \ldots, c(2007,2007)$.

It results that the maximum k of coloured cells that a charrua table can have is $2007 \times 2007 - 2007 = 2006 \times 2007$.

(b) We can choose in $2007!$ ways the uncoloured cells, one in each row (but not two in the same columns).

5. Let $ABCDE$ be a convex pentagon that satisfies the following:

(i) There is a circle Γ tangent to each of the sides.

(ii) The lengths of the sides are all whole numbers.

(iii) At least one of the sides of the pentagon has length 1.

(iv) The side AB has length 2.

Let P be the point of tangency of Γ with AB.

(a) Determine the length of segments AP and BP.

(b) Give an example of a pentagon satisfying the given conditions.

Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We print Zvonaru's solution.

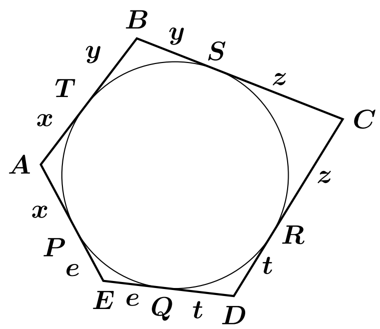
(a) Let Q, R, S, T be the points of tangency of Γ with BC, CD, DE, EA respectively.

Let $AP = AT = x$, $BP = BQ = y$, $CQ = CR = z$, $DR = DS = t$, $ES = ET = e$.

We denote by $\mathbb{Z}$ the set of integers.

Without loss of generality, we may assume that $x \geq y$.

We have



$$\left.\begin{array}{r} x+y \in \mathbb{Z} \\ y+z \in \mathbb{Z} \end{array}\right\} \Rightarrow \quad x+y-(y+z) \in \mathbb{Z} \Rightarrow x-z \in \mathbb{Z} \qquad (1)$$

$$\left.\begin{array}{r} z+t \in \mathbb{Z} \\ t+e \in \mathbb{Z} \end{array}\right\} \Rightarrow \quad z-e \in \mathbb{Z} \qquad (2)$$

$$\left.\begin{array}{r} e+x \in \mathbb{Z} \\ x+y \in \mathbb{Z} \end{array}\right\} \Rightarrow \quad e-y \in \mathbb{Z} \qquad (3)$$

By (1), (2), and (3) we obtain $x - z + e - y \in \mathbb{Z} \Rightarrow x - y \in \mathbb{Z}$. It results that there is one integer m such that

$$\begin{aligned} x - y &= m \\ x + y &= 2. \end{aligned}$$

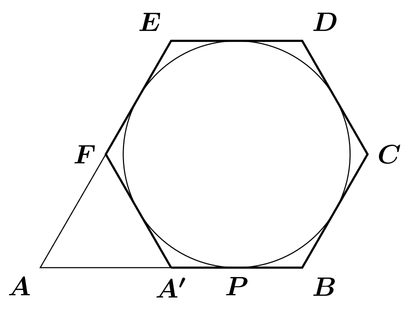
It follows that $x = 1 + \frac{m}{2}$. Since $1 \leq x < 2$, we deduce that $1 \leq 1 + \frac{m}{2} < 2 \Leftrightarrow 0 \leq m < 2$.

If $m = 0$, then $x = y = 1$ and we obtain that z, e and t are positive integers; this leads to a contradiction with condition (iii).

If $m = 1$, then $x = \frac{3}{2}$, $y = \frac{1}{2}$, hence $AP = \frac{3}{2}$, $BP = \frac{1}{2}$.

(b) Let $A'BCDEF$ be a regular hexagon with $A'B = 1$. This hexagon has an inscribed circle (tangent to each of the sides).

The line EF intersects the line $A'B$ at the point A. Since $\triangle AA'F$ is equilateral, we have $AB = AE = 2$, $BC = CD = DE = 1$, hence the pentagon $ABCDE$ satisfies the given conditions.



6. Show that for each positive whole number n, there is a positive whole number k such that the decimal representation of each of the numbers $k, 2k, \ldots, nk$ contains all the digits $0, 1, 2, \ldots, 9$.

Solved by Oliver Geupel, Brühl, NRW, Germany.

Let the decimal representation of the number k consist of the leading digit 1, followed by 99 blocks of length $\lfloor \log n \rfloor + 3$ where the m^{th} block is the decimal representation of the number $m+1$ with leading zeros ($m = 1, 2, \ldots, 99$):

$$1\ \underbrace{0\ldots 0002}_{\text{block } 1}\ \underbrace{0\ldots 0003}_{\text{block } 2} \ldots \underbrace{0\ldots 0099}_{\text{block } 98}\ \underbrace{0\ldots 0100}_{\text{block } 99}$$

Note that the length of the decimal representation of the integer $j(m+1)$ is $\lfloor \log(j(m+1)) \rfloor + 1 \le \lfloor \log n + \log 100 \rfloor + 1 \le \lfloor \log n \rfloor + 3$ $(j = 1, \ldots, n)$. The decimal representation of the number jk consists of the decimal representation of the number j followed by 99 blocks of length $\lfloor \log n \rfloor + 3$ where the m^{th} block is the decimal representation of the number $j(m+1)$ with leading zeros.

It therefore suffices to prove that, for each j ($j = 1, \ldots, n$), every digit occurs in the decimal representation of one of the numbers $j, 2j, 3j, \ldots, 100j$. To this end, for any fixed j consider the integer ℓ such that $10^{\ell-1} \le j < 10^{\ell}$. It holds $10^{\ell+1} \le 100j$ and each of the nine intervals

$$[10^l, 2 \cdot 10^l), [2 \cdot 10^l, 3 \cdot 10^l), \ldots, [9 \cdot 10^l, 10^{l+1})$$

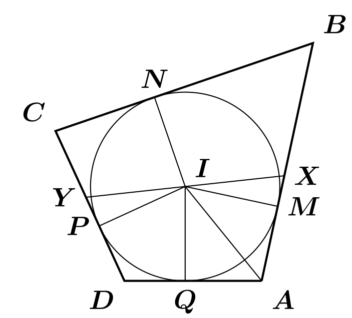
contains a number from the arithmetic sequence $j, 2j, 3j, \ldots, 100j$. The leading digit of the numbers in the m^{th} interval is m ($m = 1, 2, \ldots, 9$). Thus, the digits $1, 2, \ldots, 9$ occur. The last digit of $100j$ is zero. This completes the proof.

Next we move to the file of solutions from our readers to problems given in the September 2010 number of the *Corner* and the 2007 Bulgarian National Olympiad at [2010: 274].

1. (Emil Kolev, Alexandar Ivanov) The quadrilateral $ABCD$ is such that $\angle BAD + \angle ADC > 180^\circ$ and is circumscribed around a circle of center I. A line through I meets AB and CD at points X and Y, respectively. Prove that if $IX = IY$ then $AX \cdot DY = BX \cdot CY$.

Solved by Titu Zvonaru, Comăneşti, Romania.

Let M, N, P, Q be the projections of point I onto the sides AB, BC, CD, DA, respectively. Since $IX = IY$ and $IM = IP$, the right-angled triangles IMX and IPY are congruent, hence $\angle IXM = \angle IYP$.



If Y lies between D and P, and M lies between A and X, then $AB \| CD$, a contradiction with $\angle BAD + \angle AOC > 180^\circ$. We may assume that Y lies between C and P, and X lies between B and M.

We denote $a = AQ = AM$, $b = BM = BN$, $c = CN = CP$, $d = DP = DQ$, $\alpha = \angle BAD$, $\beta = \angle CBA$, $\gamma = \angle DCB$, $\delta = \angle ADC$, $\varphi = \angle IXM = \angle IYP$, $m = XM = YP$, $r = IM = IN = IP = IQ$.

Since $\tan\frac{\alpha}{2} = \tan\angle MAI = \frac{IM}{AM} = \frac{r}{a}$, we have

$$\varphi = \frac{360^\circ - \alpha - \delta}{2} = \frac{\beta+\gamma}{2},$$

hence

$$\tan\varphi = -\tan\frac{\alpha+\delta}{2} = -\frac{\frac{r}{a}+\frac{r}{d}}{1-\frac{r}{a}\cdot\frac{r}{d}} = \frac{r(a+d)}{r^2-ad}$$

and

$$\tan\varphi = \tan\frac{\beta+\gamma}{2} = \frac{r(b+c)}{bc-r^2}.$$

Since $\tan\varphi = \frac{r}{m}$, we deduce that

$$\frac{1}{m} = \frac{a+d}{r^2-ad} = \frac{b+c}{bc-r^2} = \frac{a+b+c+d}{bc-ad},$$

hence

$$m = \frac{bc-ad}{a+b+c+d}. \tag{1}$$

It follows that $AX \cdot DY = BX \cdot CY$ is equivalent to

$$\begin{aligned}(a+m)(d+m) &= (b-m)(c-m)\\ ad + m(a+d) &= bc - m(b+c)\\ m &= \frac{bc-ad}{a+b+c+d},\end{aligned}$$

which is true by (1).

3. (Nikolai Nikolov, Oleg Mushkarov) Find the least natural number n for which $\cos\frac{\pi}{n}$ cannot be expressed in the form $p+\sqrt{q}+\sqrt[3]{r}$, where p, q and r are rational numbers.

Solved by Mohammed Aassila, Strasbourg, France.

We will prove that the least positive integer is $n = 7$. Let $\omega = e^{\frac{i\pi}{7}}$. We have $0 = \omega^7 + 1 = (\omega+1)(1-\omega+\omega^2-\cdots+\omega^6)$, hence $0 = (\omega^3+\omega^{-3}) - (\omega^2+\omega^{-2}) + (\omega+\omega) - 1$. Let $x = \omega+\omega^{-1} = 2\cos\frac{\pi}{n}$, then x verifies the equation $0 = (x^3-3x)-(x^2-2)+x-1 = x^3-x^2-2x+1$. Let $P(x) = x^3-x^2-2x+1$. Note that the roots of P are $2\cos(\frac{\pi}{7})$, $2\cos(\frac{3\pi}{7})$ and $2\cos(\frac{5\pi}{7})$. Assume, by contradiction, that $x = P+\sqrt{q}+\sqrt[3]{r}$, then x is irrational since if $x = \frac{a}{b}$ with $\gcd(a,b) = 1$ then $a^3-a^2b-2ab^2+b^3 = 0$, hence $b \mid a^3$, and $b = \pm 1$ and similarly $a = \pm 1$, in conclusion $x = \pm 1$, which is false.

Case 1: $r = 0$.

Then $x^2 - 2px + p^2 - q = 0$. By the Euclidean division we have $P(X) = (X - a)(X^2 - 2pX + p^2 - q) + (bX + c)$. Hence $bx + c = 0$. Since x is irrational, we have $b = c = 0$ and hence $P(X) = (X - a)(X^2 - 2pX + p^2 - q)$. Consequently, P has a rational root a. Impossible.

Case 2: $q = 0$.

We have $(x - p)^3 = r$. Since $X^3 - 3pX^2 + 3p^2X - p^3 - r$ is not proportional to $P(X)$ (compare the coefficients of X and of X^2), an Euclidean division yields that x is a root of a polynomial (not equal to 0) and with degree ≤ 2, hence we are in case 1.

Case 3: $q \neq 0$ and $r \neq 0$.

We have $r = (x - p - \sqrt{q})^3$. Like in case 2, there exists a polynomial $A \neq 0$ of degree ≤ 2 in $\mathbb{Q}[\sqrt{q}][X]$ such that $A(x) = 0$. If $\deg A = 1$ then we are in case 1. If $\deg A = 2$ by an Euclidean division we have $P(X) = A(X)(X - a) + B(X)$. Since $B(x) = 0$, then thanks to case 1 we have $B = 0$, hence P has a for a root in $\mathbb{Q} + \mathbb{Q}\sqrt{q}$. As in the first case, we have a contradiction.

Alternative solution: (in french and using higher algebra).

Soit $x = \cos(\pi/7)$. Soit $K = \mathbb{Q}(x)$. Alors K est une extension galoisienne de degré $\varphi(14)/2 = 3$ de $\mathbb{Q}$. Si x était de la forme $p + \sqrt{q} + \sqrt[3]{r}$, alors en élevant au cube $x - p - \sqrt{q}$, on voit que $\sqrt{q} \in K$. Comme $[K : \mathbb{Q}]$ est impair, il ne peut pas contenir d'extension quadratique de $\mathbb{Q}$ donc $\sqrt{q}$ est rationnel. Par conséquent, x est de la forme $p + y$, où $y = \sqrt[3]{r}$. Comme $K/\mathbb{Q}$ est galoisienne, K contient jy. On vérifie ensuite que $K = \mathbb{Q}[j, y]$ est de degré 6 sur $\mathbb{Q}$. Contradiction.

4. ((Emil Kolev, Alexandar Ivanov) Let $k > 1$ be an integer. A set of positive integers S is called *good* if all positive integers can be painted in k colors such that no element of S is a sum of two distinct numbers of the same color. Find the largest positive integer t for which the set

$$S = \{a + 1, a + 2, a + 3, \ldots, a + t\}$$

is good for all positive integers a.

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We claim that the largest such t, which we denote $t_{\max}(k)$, is $2k - 2$.

Case 1.

To show that $t_{\max}(k) \geq 2k - 2$, consider even and odd a separately.

(a) If $a = 2r - 1$ and $t = 2k - 2$, then

$$S = \{2r, 2r + 1, \ldots, 2r + 2k - 3\}.$$

We assign colours as follows:

Colour 1	$1, 2, 3, \ldots, r$
Colour 2	$r + 1$
Colour 3	$r + 2$
$\vdots$	
Colour $k - 1$	$r + k - 2$
Colour k	$r + k - 1$, $r + k$, $r + k + 1, \ldots, 2r + 2k - 4$

All other positive integers are assigned colour k. The sum of any two distinct integers of colour 1 is smaller than any element of S; the sum of any two distinct integers of colour k is larger than any element of S. Thus, no element of S is a sum of two distinct integers of the same colour.

(b) If $a = 2r$ and $t = 2k - 2$, then

$$S = \{2r + 1, 2r + 2, \ldots, 2r + 2k - 2\}.$$

We assign colours as before except that colour k is assigned to all integers from $n + k - 1$ through $2r + 2k - 3$. Again, no element of S is the sum of two distinct integers of the same colour.

Case 2. To show that $t_{\max}(k) \le 2k - 2$, suppose $t = 2k - 1$, and set $a = 2$. Then $S = \{3, 4, 5, \ldots, 2k + 1\}$. Consider the colours assigned to the integers $1, 2, 3, \ldots, k+1$. Suppose i and j are elements of this list with $i < j$, and assume i and j are assigned the same colour. Then $3 \le i + j \le k + (k + 1) = 2k + 1$, so $i + j \in S$. Hence the set S is not good.

Thus, the largest t is given by $t_{\max}(k) = 2k - 2$.

5. (Oleg Mushkarov, Nikolai Nikolov) Find the least number m for which any five equilateral triangles with combined area m can cover an equilateral triangle of area 1.

Solved by George Apostolopoulos, Messolonghi, Greece; and Oliver Geupel, Brühl, NRW, Germany. We give the version of Geupel.

The answer is $m = 2$. For convenience of presentation, we rescale the given equilateral triangle T such that its side length is 1.

We firstly prove that any five equilateral triangles $T_1, \ldots, T_5$ with side lengths $a_1 \le \cdots \le a_5$ and $a_1^2 + \cdots + a_5^2 = 2$ can cover T.

Since the case $a_5 \ge 1$ is obvious, we may assume $a_5 < 1$. Then

$$(a_3 + a_4)^2 \ge 3a_3^2 + a_4^2 \ge 2 - a_5^2 > 1\,.$$

Let us place T_3, T_4, T_5 in one of the angles of T each, as shown in figure 1. If T_3, T_4, and T_5 cover T then we are done. Otherwise, there remains an uncovered equilateral triangle T' in the interior of T. Each point of T is covered by at most two of the triangles T_3, T_4, T_5, T'. Hence, the combined area of T_1 and T_2 is not less than twice the area of T'; thus T_2 covers T'. We have proved that $m \le 2$.

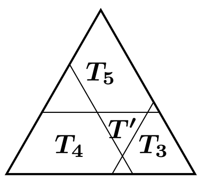


Figure 1

In order to show that $m \geq 2$, for arbitrary $\epsilon > 0$ we present equilateral triangles $T_1, \ldots, T_5$ with side lengths $a_1 \leq \cdots \leq a_5$ such that $a_1^2 + \cdots + a_5^2 > 2 - \epsilon$ that cannot cover T. Let

$$\delta = \frac{\epsilon}{20}, \qquad a_1 = a_2 = a_3 = \delta, \qquad a_4 = a_5 = 1 - 5\delta.$$

Indeed,

$$a_1^2 + \cdots + a_5^2 > 2 - 20\delta = 2 - \epsilon.$$

Let us pin congruent equilateral triangles U, V, W with side length 2δ to the vertices of T as shown in figure 2. On applying T_3 and T_4 to T, each of T_3 and T_4 can meet at most one of the triangles U, V, W. Hence, there is a triangle among U, V, W which is not met by T_3 and T_4, say U has this property. But the combined area of T_1, T_2, T_3 is less than the area of U, so T_1, T_2, T_3 cannot cover U. This completes both our counterexample and the proof that $m \geq 2$.

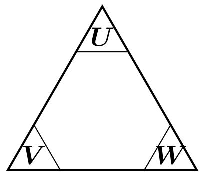


Figure 2

Next we look at solutions for the 48$^{\text{th}}$ IMO Bulgarian Team, First Selection Test, given at [2010: 275].

1. The sequence $\{a_i\}_{i=1}^{\infty}$ is such that $a_1 > 0$ and $a_{n+1} = \frac{a_n}{1+a_n^2}$ for $n \geq 1$.

(a) Prove that $a_n \leq \frac{1}{\sqrt{2n}}$ for $n \geq 2$;

(b) Prove that there exists n such that $a_n > \frac{7}{10\sqrt{n}}$.

Solved by Arkady Alt, San Jose, CA, USA; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA. We give the solution by Alt.

Since $a_n > 0, n \geq 1$ then

$$\begin{aligned} a_{n+1} = \frac{a_n}{1+a_n^2} &\iff \frac{1}{a_{n+1}^2} = \left(\frac{1}{a_n} + a_n\right)^2 \\ &\iff \frac{1}{a_{n+1}^2} = \frac{1}{a_n^2} + a_n^2 + 2 \iff b_{n+1} = b_n + 2 + \frac{1}{b_n}, \end{aligned}$$

where $b_n := \dfrac{1}{a_n^2}$, $n \geq 1$ and we will prove:

a) $b_n \geq 2n$ for $n > 2$;

b) There is n such that $b_n < \frac{100n}{49}$.

a) Since $b_{n+1} = b_n + 2 + \frac{1}{b_n} > b_n + 2$ and $b_2 \geq 4$, by induction $b_n \geq 2n$.

b) Since $b_n \geq 2n$ for $n \geq 2$ then $b_{n+1} = b_n + 2 + \dfrac{1}{b_n} \leq b_n + 2 + \dfrac{1}{2n}$, $n \geq 2$ and, therefore,

$$b_{n+1} - b_2 = \sum_{k=2}^{n} (b_{k+1} - b_k) \leq \sum_{k=2}^{n} \left(2 + \frac{1}{2n}\right) = 2(n-1) + \frac{1}{2}(h_n - 1)\ ,$$

where $h_n = \sum\limits_{k=1}^{n} \dfrac{1}{n}$. Thus,

$$b_{n+1} \leq 2n - \frac{5}{2} + \frac{1}{2}h_n + b_2 < 2(n+1) + \frac{1}{2}h_{n+1} + b_2,\ n \geq 2$$
$$\Longrightarrow b_n < 2n + \frac{1}{2}h_n + b_2,\ n \geq 3\,.$$

Note that $h_n < \sqrt{2n}$, $n \in \mathbb{N}$. Indeed, by the Cauchy Inequality we have

$$h_n^2 \leq n \cdot \sum_{k=1}^{n} \frac{1}{k^2}$$

and

$$\sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{(k-1)\,k} = 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 + 1 - \frac{1}{n} < 2\,.$$

Since $\frac{100n}{49} = 2n + \frac{2n}{49}$ and $h_n < \sqrt{2n}$ then it suffices to prove that there is n such that $\frac{1}{2}\sqrt{2n} + b_2 < \frac{2n}{49} \iff 49b_2 < \sqrt{2n}\left(\sqrt{2n} - \frac{49}{2}\right)$.

It is easy to see that the latter inequality holds for any

$$n \geq\ n_0 = \max\left\{\frac{49^2 b_2^2}{2}, \frac{51^2}{8}\right\}\ .$$

Another variant of ending solution (b):

Since $2n < b_n < 2n + \dfrac{1}{2}\sqrt{2n} + b_2$ and $\lim\limits_{n\to\infty} \dfrac{2n + \frac{1}{2}\sqrt{2n} + b_2}{n} = 2$ then $\lim\limits_{n\to\infty} \dfrac{b_n}{n} = 2$ and, therefore, for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\dfrac{b_n}{n} < 2 + \varepsilon \iff b_n < (2+\varepsilon)\,n$ for all $n > n_0(\varepsilon)$. In particular for $\varepsilon = \dfrac{2}{49}$ we have $b_n < \left(2 + \dfrac{2}{49}\right) n = \dfrac{100n}{49}$ for all $n > n_0\left(\dfrac{2}{49}\right)$.

That completes the *Corner* for this issue.