BOOK REVIEWS

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Lobachevski Revisited by Seth Braver
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During the 1820s Lobachevski (1792-1856) in Russia and Bolyai (1802-1860) in Hungary independently discovered non-Euclidean Geometry—the geometry in which there are two lines parallel to a given line through a given point not on that line. Because of the overwhelming importance of their ideas, it might be hard for us today to understand how their work could have been so thoroughly ignored by their contemporaries—it was only after their deaths that the mathematical world paid much attention to the subject, and several decades elapsed before the full implication of their achievements became appreciated. Whereas Bolyai was so discouraged that he gave up publishing mathematics, Lobachevski optimistically produced further Russian accounts of non-Euclidean geometry; when his fellow Russians failed to recognize the significance of his work, he then published treatments of his theory in French and in German. His little German book of 1840, *Geometrische Untersuchungen zur Theorie der Parallellinien*, forms the core of the book under review here. Seth Braver's translation, with the title shortened to *The Theory of Parallels*, appears twice: at the end as a 23-page appendix, and spread out over the first 200 pages, printed in red type and supplemented by Braver's introduction and notes, printed in black. The resulting "illuminated" Lobachevski is intended for "student, professional, [and] layman."

Braver's book would certainly make a superb textbook for an undergraduate course in non-Euclidean geometry. His commentary provides the historic and philosophical background that explains the mathematical environment in which Lobachevski worked, as well as the significance of his work. His mathematical ideas are clearly motivated, and the relevant achievements of predecessors and contemporaries are briefly outlined. Explanations are provided to fill in details that many of today's students might otherwise find difficult; the commentary is informative and entertaining, which most students would appreciate. For example, when I taught such a course a few years back I was surprised when several students complained about the diagrams: in *imaginary geometry* (to use Lobachevski's terminology), straight lines in diagrams are often represented by curves so as to avoid unwanted intersections. Braver points out that although space in imaginary geometry looks the same at every point, "it looks very different at different *scales*. On a tiny scale, it resembles Euclidean geometry, and serious deviations become noticeable only on a large, possibly astronomical, scale. Since similar figures do not exist in imaginary geometry, accurate scaled down drawings are impossible." I wish I had thought of this explanation to give my students. In general, the author provides the student with good explanations of what is the same and what is different in this new geometry. He supplements Lobachevski's proofs with further details and alternative arguments, together with related results and elegant arguments from Saccheri, Lambert, Legendre, Gauss, Bolyai, and others. A teacher using this book as a text, however, would have to provide details of Euclidean theorems and proofs, and perhaps examples of modern arguments involving betweenness axioms. Also, the book comes with no exercises. As usual, I disagree with the MAA's pricing policy: the $90 US price tag on the printed edition seems designed to encourage the student to purchase the $35 e-book. According to the book representative who sent me the printed version, the electronic version is equally hard to navigate—the references are not linked so that there is no quick way to locate an item that has been cross-referenced.

The book is less successful in its effort to reach the professional. Lobachevski's intended audience was the professional mathematician of the mid-nineteenth century. He wrote quite well; his arguments provide pretty much the same level of detail that would be expected by readers of the geometry problems in **Crux**, so not much help would be needed for any reader familiar with Euclid. The supplementary comments on philosophy and history have been taken almost entirely from standard sources that are readily available and are not entirely reliable. Whereas the light, whimsical tone of the commentary might be suitable for the undergraduate seeing the material for the first time, I found many explanations lacking in depth. This shortcoming was most evident in the notes accompanying the preface where Lobachevski lists his preliminary theorems and claims that their "proofs present no difficulties." Braver is content to denigrate the first five propositions ("A Rough Start", "[they] should be demoted to the status of descriptions (or axioms)", "... if either he or Euclid had tried to prove it rigorously, they would have found the task impossible", and on and on. Far from thinking Lobachevski's arguments were faulty, I was struck by how much thought he put into the foundations, and how far his thoughts had advanced beyond Euclid. Compare Euclid's second postulate, "A finite straight line may be extended continuously in a straight line." with Lobachevski's Paragraph 3, "By extending both sides of a straight line sufficiently far, it will break out of any bounded region. In particular, it will separate a bounded plane region into two parts." Surely, he wanted to make it clear that lines stretch to infinity in both directions (which is not clear in Euclid, who demands only that his segments be extendable). We can also see here a primitive notion of separation decades before Pasch made the concept rigorous (in 1882). How more valuable it would have been had Braver discussed where the initial five "theorems" originated, why Lobachevski thought they were easily established and, most importantly, where and how he used them in subsequent proofs. At any rate, the reader does not have to be reminded every time Proposition 3 is invoked, that Lobachevski lacked 20^{th} century tools.

The author fails to mention why he felt the need for a new translation from the original German, which seems to be only superficially different from Bruce Halsted's 1881 translation. But whichever translation you can get your hands on, any person who likes geometry should read *The Theory of Parallels*—the final half-dozen propositions constitute some of the most clever and exciting mathematics ever conceived. That is where Lobachevski develops the geometry of the horosphere and then establishes that the geometry of the sphere is independent of the parallel postulate (that is, the geometry and trigonometry of a sphere is the same in imaginary geometry as in Euclidean geometry); these results lead him directly to the key formulas of imaginary geometry.

Unsolved Crux Problems

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Recently, Chris Fisher published a list of unsolved problems from *Crux*[2010 : 545, 547]. Below is a sample of two of these unsolved problems:

342$\star$. [1978 : 133, 297; 1980 : 319-22] *Proposed by James Gary Propp, Great Neek, NY, USA.*

For fixed $n \geq 2$, the set of all positive integers is partitioned into the (disjoint) subsets S_1, S_2, $\ldots$, S_n as follows: for each positive integer m, we have $m \in S_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the n subsets.

Prove that $m \in S_n$ for all sufficiently large m. (If $n = 2$, this is essentially equivalent to Problem 226 [1977 : 205]).

1754$\star$. [1992 : 175; 1993 : 151-2; 1994 : 196-9; 1995 : 236-8] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n and k be positive integers such that $2 \leq k < n$, and let x_1, x_2, $\ldots$, x_n be non-negative real numbers satisfying $\sum_{i=1}^{n} x_i = 1$. Prove or disprove that

$$\sum x_1 x_2 \ldots x_k \;\leq\; \max\left\{\frac{1}{k^k}, \frac{1}{n^{k-1}}\right\},$$

where the sum is cyclic over x_1, x_2, $\ldots$, x_n. [The case $k = 2$ is known — see inequality (1) in the solution of ***CRUX* 1662** [1992 : 188].]

Good luck solving these problems. We would love to receive your solutions so that we could cross them off our list.