

# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff member is Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON). The editor thanks Rachael Verbruggen, University of Waterloo, for her assistance with this month's solutions.

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## Mayhem Problems

*Please send your solutions to the problems in this edition by 15 February 2011. Solutions received after this date will only be considered if there is time before publication of the solutions.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.*

*The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.*

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**M457.** *Proposed by the Mayhem Staff.*

Suppose that  $A$  is a digit between 0 and 9, inclusive, and that the tens digit of the product of  $2A7$  and  $39$  is 9. Determine the digit  $A$ .

**M458.** *Proposed by the Mayhem Staff.*

Convex quadrilateral  $ABCD$  has  $AB = AD = 10$  and  $BC = CD$ . Also,  $AC$  is perpendicular to  $BD$ , with  $AC$  and  $BD$  intersecting at  $P$ . If  $BP = 8$  and  $CD = CP + 2$ , determine the area of quadrilateral  $ABCD$ .

**M459.** *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine whether or not it is possible to create a collection of ten distinct subsets of  $S = \{1, 2, 3, 4, 5, 6\}$  so that each subset contains three elements, each element of  $S$  appears in five subsets, and each pair of elements from  $S$  appears in two subsets.

**M460.** *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let  $a$  and  $b$  be positive real numbers. Define  $A = \frac{a+b}{2}$ ,  $G = \sqrt{ab}$ , and  $K = \sqrt{\frac{a^2+b^2}{2}}$ . Prove that (a)  $G^2 + K^2 = 2A^2$ , (b)  $A^2 \geq KG$ , (c)  $G + K \leq 2A$ , and (d)  $G^4 + K^4 \geq 2A^4$ .

**M461.** *Proposed by Landelino Arboniés, Colegio Marcelino Champagnat, Santo Domingo, Dominican Republic.*

A *Champagnat* number is equal to the sum of all the digits in a set of consecutive positive integers, one of which is the number itself. Thus, 42 is a *Champagnat* number, since 42 is the sum of all of the digits of 39, 40, 41, 42, 43, 44. Prove that there exist infinitely many *Champagnat* numbers.

**M462.** *Proposed by Alex Song, Detroit Country Day School, Detroit, MI, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let  $\lfloor x \rfloor$  denote the greatest integer not exceeding  $x$  and let  $\lceil x \rceil$  denote the smallest integer greater than or equal to  $x$ . For example,  $\lfloor 3.1 \rfloor = 3$ ,  $\lceil 3.1 \rceil = 4$ ,  $\lfloor -1.4 \rfloor = -2$ , and  $\lceil -1.4 \rceil = -1$ . Determine all real numbers  $x$  for which  $\lfloor x \rfloor \lceil x \rceil = x^2$ .

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**M457.** *Proposé par l'Équipe de Mayhem.*

On suppose que  $A$  est un chiffre entre 0 et 9 inclusivement, et que le chiffre des dizaines du produit  $2A7$  et  $39$  est 9. Trouver  $A$ .

**M458.** *Proposé par l'Équipe de Mayhem.*

Soit  $ABCD$  un quadrilatère convexe avec  $AB = AD = 10$  et  $BC = CD$ . De plus, soit  $AC$  perpendiculaire à  $BD$  et  $P$  leur point d'intersection. Si  $BP = 8$  et  $CD = CP + 2$ , trouver l'aire du quadrilatère  $ABCD$ .

**M459.** *Proposé par Neven Jurič, Zagreb, Croatie.*

Déterminer si oui ou non, il est possible de créer une collection de dix sous-ensembles distincts de  $S = \{1, 2, 3, 4, 5, 6\}$  de sorte que chaque sous-ensemble contienne trois éléments, que chaque élément de  $S$  apparaisse dans cinq sous-ensembles, et que chaque paire d'éléments de  $S$  apparaisse dans deux sous-ensembles.

**M460.** *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit  $a$  et  $b$  deux nombres réels positifs. On définit  $A = \frac{a+b}{2}$ ,  $G = \sqrt{ab}$  et  $K = \sqrt{\frac{a^2+b^2}{2}}$ . Montrer que (a)  $G^2 + K^2 = 2A^2$ , (b)  $A^2 \geq KG$ , (c)  $G + K \leq 2A$ , et (d)  $G^4 + K^4 \geq 2A^4$ .

**M461.** *Proposé par Landelino Arboniés, Colegio Marcelino Champagnat, Santo Domingo, République dominicaine.*

Un nombre de *Champagnat*  $n$  est défini comme la somme de tous les chiffres d'une suite d'entiers consécutifs comprenant  $n$ . Ainsi, 42 est un nombre de *Champagnat* puisqu'il est dans la suite 39, 40, 41, 42, 43, 44. Montrer qu'il existe une infinité de nombres de *Champagnat*.

**M462.** *Proposé par Alex Song, Detroit Country Day School, Detroit, MI, É-U et Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

On désigne par  $\lfloor x \rfloor$  le plus grand entier n'excédant pas  $x$  et par  $\lceil x \rceil$  le plus petit entier plus grand ou égal à  $x$ . Par exemple,  $\lfloor 3.1 \rfloor = 3$ ,  $\lceil 3.1 \rceil = 4$ ,  $\lfloor -1.4 \rfloor = -2$ , et  $\lceil -1.4 \rceil = -1$ . Déterminer tous les nombres réels  $x$  pour lesquels  $\lfloor x \rfloor \lceil x \rceil = x^2$ .

## Mayhem Solutions

**M420.** *Proposed by the Mayhem Staff.*

Riley is a poor starving university student, but is mathematically astute. He notices that five suppers in residence cost the same as seven lunches. After one week of skipping supper most nights, he notices that five lunches and one supper cost \$48 in total. How much do 16 suppers cost?

*Solution by Oscar Xia, student, St. George's School, Vancouver, BC.*

Let one lunch cost  $x$  dollars and one supper cost  $y$  dollars.

Since five suppers cost the same as seven lunches, then  $5y = 7x$ . Since five suppers and one lunch cost \$48, then  $5x + y = 48$ .

From the first equation,  $x = \frac{5}{7}y$  and so we obtain  $5\left(\frac{5}{7}y\right) + y = 48$ , or  $\frac{32}{7}y = 48$ , or  $y = \frac{21}{2}$ .

Therefore, 16 suppers cost  $16\left(\frac{21}{2}\right) = 168$  dollars.

*Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; LEWIS HUGHES, Auburn University, Montgomery, AL, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Perú, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEIRO, Viveiro, Spain; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.*

**M421.** *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let  $\lfloor x \rfloor$  be the greatest integer less than or equal to the real number  $x$ . Determine all real numbers  $x$  such that

$$\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{3}{x} \right\rfloor = 4.$$

*Solution by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA.*

Since  $x$  cannot be 0, then we let  $y = \frac{1}{x}$ . Therefore, we are looking for all real nonzero solutions to  $\lfloor y \rfloor + \lfloor 3y \rfloor = 4$ . Note that  $y$  cannot be

negative, because if it were, then the left side of the equation would be negative. Therefore,  $y$  is positive.

If  $y < 1$ , then  $3y < 3$  and so  $\lfloor y \rfloor + \lfloor 3y \rfloor < y + 3y < 1 + 3 = 4$ . Therefore,  $y \geq 1$ .

If  $y \geq \frac{4}{3}$ , then  $3y \geq 4$ . In this case,  $\lfloor y \rfloor \geq 1$  and  $\lfloor 3y \rfloor \geq 4$ , so  $\lfloor y \rfloor + \lfloor 3y \rfloor \geq 5$ . Therefore,  $y < \frac{4}{3}$ .

Combining what we know, we have  $1 \leq y < \frac{4}{3}$ . For these  $y$ , we have  $\lfloor y \rfloor = 1$ . Also, we have  $3 \leq 3y < 4$  and so  $\lfloor 3y \rfloor = 3$ . Therefore, for all of these  $y$ , we have  $\lfloor y \rfloor + \lfloor 3y \rfloor = 4$ .

Therefore, the values of  $y$  that satisfy the equation are  $1 \leq y < \frac{4}{3}$ , and so the values of  $x$  that satisfy the original equation are  $\frac{3}{4} < x \leq 1$ .

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILLI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; OSCAR XIA, student, St. George's School, Vancouver, BC; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. One incorrect solution and two incomplete solutions were submitted.

**M422.** Proposed by Adnan Arapovic, student, University of Waterloo, Waterloo, ON.

Prove that

$$\sum_{k=1}^n \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Since  $\binom{k+1}{2} = \frac{k(k+1)}{2}$ , then the given equation is equivalent to

$$\sum_{k=1}^n \binom{k+1}{2} = \binom{n+2}{3}.$$

The total number of 3-element subsets of  $\{1, 2, 3, \dots, n+1, n+2\}$  is equal to  $\binom{n+2}{3}$ .

These subsets can also be counted in the following way. First, choose the largest element,  $M$ , in the subset. Note that  $M$  can take any value from 3 to  $n+2$ . Next, choose the remaining two elements from among the numbers 1 to  $M-1$ . Let  $M = k+2$ , where  $k = 1, 2, \dots, n$ . For a fixed value of  $k$ , there are  $\binom{k+1}{2}$  ways of choosing the remaining two elements. Summing over all possible values of  $k$ , we see that the total number of 3-element subsets of  $\{1, 2, 3, \dots, n+1, n+2\}$  is also equal to  $\sum_{k=1}^n \binom{k+1}{2}$ .

Therefore,  $\sum_{k=1}^n \binom{k+1}{2} = \binom{n+2}{3}$ .

Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Perú, Lima, Peru; KONSTANTINOS AL. NAKOS, Agrinio, Greece; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON (second and third solutions); JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; OSCAR XIA, student, St. George's School, Vancouver, BC; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Three incomplete solutions were submitted.

**M423.** Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.

The tens digit of a perfect square  $S$  is three greater than the ones digit of  $S$ . Determine all possible remainders when  $S$  is divided by 100.

*Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.*

Every perfect square has its units digit in the set  $\{0, 1, 4, 5, 6, 9\}$ . (This can be seen by squaring the integers from 0 to 9 and observing that the pattern of units digits continues for larger squares.)

Since the tens digit is to be 3 greater than the units digit, then units digit cannot be greater than 6, so the only possible remainders when  $S$  is divided by 100 (that is, the possible pairs of last two digits of  $S$ ) are 30, 41, 74, 85, and 96. That is,  $S$  must have one of the following forms

$$100k + 30, \quad 100k + 41, \quad 100k + 74, \quad 100k + 85, \quad 100k + 96$$

for some integer  $k$ .

Note that even perfect squares must be divisible by 4, because an even perfect square is the square of an even integer, say  $2n$  for some integer  $n$ , so the square is  $4n^2$ . Any integer of the form  $100k + 30$  or  $100k + 74$  is not divisible by 4, so cannot be a perfect square. Thus,  $S$  cannot end in 30 or 74.

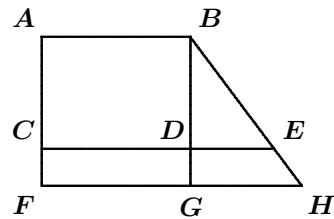
Also,  $S$  cannot end in 85. If  $S$  did end in 85, then its square root would end in 5, so  $S$  would be of the form  $(10a + 5)^2$  for some integer  $a$ . In this case,  $S = 100a^2 + 100a + 25$ , which ends in 25. So  $S$  cannot end in 85.

Therefore, the only possibilities for the last two digits of  $S$  are 41 and 96. Each of them is possible since  $21^2 = 441$  and  $14^2 = 196$ .

Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain. One incomplete solution and one incorrect solution were submitted.

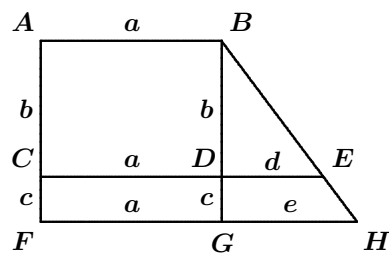
**M424.** Proposed by Margo Kondratieva, Memorial University of Newfoundland, St. John's, NL.

In the diagram, line segments  $AB$ ,  $CDE$ , and  $FGH$  are parallel. Also, line segments  $ACF$  and  $BDG$  are perpendicular to  $AB$ . Suppose that the area of rectangle  $ABDC$  is  $x$ , the area of rectangle  $CDGF$  is  $y$ , and the area of  $\triangle BDE$  is  $z$ . Determine the area of  $DEHG$  in terms of  $x$ ,  $y$ , and  $z$ .



*Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.*

Since line segments  $AB$ ,  $CDE$ , and  $FGH$  are parallel, and line segments  $ACF$  and  $BDG$  are perpendicular to  $AB$ , then  $AB$ ,  $CD$ , and  $FGH$  are each perpendicular to both  $ACF$  and  $BDG$ . Therefore,  $ABDC$  and  $CDGF$  are rectangles, and  $\triangle BDE$  is similar to  $\triangle BGH$ , since they share a common angle at  $B$ .



Suppose that  $AB = CD = FG = a$ ,  $AC = BD = b$ ,  $CF = DG = c$ ,  $DE = d$ , and  $GH = e$ .

Since  $\triangle BDE$  is similar to  $\triangle BGH$ , we then have  $\frac{DE}{GH} = \frac{BD}{BG}$ , and so  $\frac{d}{e} = \frac{b}{b+c}$ , which gives  $e = \frac{d(b+c)}{b}$ .

Since the area of  $ABDC$  is  $x$ , then  $ab = x$ . Since the area of  $CDGF$  is  $y$ , then  $ac = y$ . Since the area of  $\triangle BDE$  is  $z$ , then  $bd = 2z$ .

From the first two equations,  $\frac{x}{y} = \frac{ab}{ac} = \frac{b}{c}$ . From the second and third equations,  $abcd = 2yz$ , which gives  $cd = \frac{2yz}{ab} = \frac{2yz}{x}$ .

Since  $DEHG$  is a trapezoid, then

$$\begin{aligned} \text{Area of } DEHG &= \frac{1}{2}(d+e)c = \frac{1}{2}cd + \frac{1}{2}ce = \frac{1}{2}cd + \frac{1}{2}c \cdot \frac{d(b+c)}{b} \\ &= \frac{1}{2}cd + \frac{1}{2}cd + \frac{1}{2} \cdot \frac{dc^2}{b} = cd + \frac{c^2d}{b} = cd + \frac{1}{2}cd \cdot \frac{c}{b} \\ &= \frac{2yz}{x} + \frac{1}{2} \cdot \frac{2yz}{x} \cdot \frac{y}{x} = \frac{2xyz + y^2z}{x^2}. \end{aligned}$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; G.C. GREUBEL, Newport News, VA, USA; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Perú, Lima, Peru; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES

“Abastos”, Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. One incorrect solution was submitted.

**M425.** Proposed by Titu Zvonaru, Comănești, Romania.

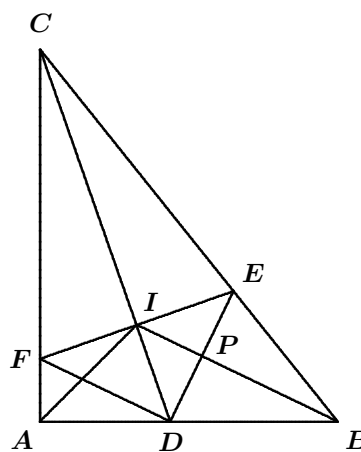
In  $\triangle ABC$ ,  $\angle BAC = 90^\circ$  and  $I$  is the incentre. The interior bisector of angle  $C$  meets  $AB$  at  $D$ . The line segment through  $D$  perpendicular to  $BI$  intersects  $BC$  at  $E$ . The line segment through  $D$  parallel to  $BI$  meets  $AC$  at  $F$ . Prove that  $E$ ,  $I$ , and  $F$  are collinear.

*Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Since  $\triangle ABC$  is right-angled at  $A$ , then  $\angle ABC + \angle ACB = 90^\circ$ .

Consider  $\triangle CIB$ . We see that  $\angle DIB$  is an external angle of this triangle, so  $\angle DIB = \angle IBC + \angle ICB$ . Since  $CI$  and  $BI$  bisect  $\angle ACB$  and  $\angle ABC$ , respectively, then  $\angle IBC = \frac{1}{2}\angle ABC$  and  $\angle ICB = \frac{1}{2}\angle ACB$ . Therefore,  $\angle DIB = \frac{1}{2}(\angle ABC + \angle ACB) = \frac{1}{2}(90^\circ) = 45^\circ$ .

Since  $DF$  is parallel to  $BI$ , then  $\angle FDI = \angle DIB = 45^\circ$ . Since  $AI$  bisects  $\angle BAC$ , then  $\angle FAI = \angle DAI = 45^\circ$ . Thus,  $FI$  subtends equal angles at  $A$  and at  $D$ . Therefore, quadrilateral  $FADI$  is cyclic.



In addition, chord  $DI$  in cyclic quadrilateral subtends both  $\angle DAI$  and  $\angle DFI$ . Thus,  $\angle DFI = \angle DAI = 45^\circ$ . We then see that in  $\triangle FID$ , there are two  $45^\circ$  angles, so  $\angle FID = 90^\circ$ .

Suppose that  $BI$  and  $DE$  intersect at  $P$ . Since  $BI$  and  $DE$  are perpendicular and  $\angle EBP = \angle DBP$  (because  $BI$  is an angle bisector) and  $BP$  is a common side in  $\triangle EBP$  and  $\triangle DBP$ , then these two triangles are congruent. Therefore,  $DP = EP$ .

This tells us that  $\triangle IDP$  and  $\triangle IEP$  are congruent, since the triangles have the side  $IP$  in common,  $DP = EP$ , and  $\angle IPD = \angle IPE = 90^\circ$ . Therefore,  $\angle DIP = \angle EIP$ .

Thus,  $\angle DIE = 2\angle DIP = 2\angle DIB = 90^\circ$ .

Finally,  $\angle FIE = \angle FID + \angle DIE = 90^\circ + 90^\circ = 180^\circ$ . This tells us that  $E$ ,  $I$ , and  $F$  are collinear, as required.

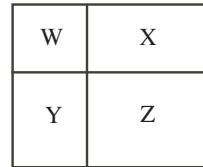
Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; RICARD PEIRÓ, IES “Abastos”, Valencia, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Three incorrect solutions were submitted.

## Problem of the Month

Ian VanderBurgh

Some problems have an algebraic solution for those who naturally like to convert things to algebra. However, some of us don't naturally immediately think to use algebra, and so it's nice to try to find solutions that are less algebraic. Often this latter type of solution can provide a bit more insight into what is actually going on.

**Problem 1** (2010 Fermat Contest) A rectangle is divided into four smaller rectangles, labelled  $W$ ,  $X$ ,  $Y$ , and  $Z$ , as shown. The perimeters of rectangles  $W$ ,  $X$ , and  $Y$  are 2, 3, and 5, respectively. What is the perimeter of rectangle  $Z$ ?



Before seeing this problem, I had seen similar problems involving areas where we are given the areas of three of the sections and asked to find the area of the fourth. I don't remember ever having seen such a problem involving perimeters. Here is an algebraic solution to this problem.

**Solution to Problem 1.** Label the lengths of the relevant vertical and horizontal segments as  $a$ ,  $b$ ,  $c$ , and  $d$ , as in the diagram.

Rectangle  $W$  is  $b$  by  $c$ , so it has perimeter  $2b + 2c$ . This equals 2. Rectangle  $X$  is  $b$  by  $d$ , so its perimeter is  $2b + 2d$ . This equals 3. Rectangle  $Y$  is  $a$  by  $c$ , so its perimeter is  $2a + 2c$ . This equals 5.

Rectangle  $Z$  is  $a$  by  $d$ , so its perimeter is  $2a + 2d$ . Since  $2b + 2c = 2$  and  $2b + 2d = 3$  and  $2a + 2c = 5$ , then

$$\begin{aligned}
 2a + 2d &= (2a + 2b + 2c + 2d) - (2b + 2c) \\
 &= (2a + 2c) + (2b + 2d) - (2b + 2c) \\
 &= 5 + 3 - 2 = 6.
 \end{aligned}$$

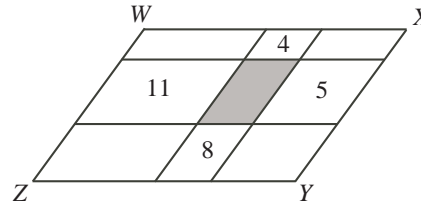
■

This solution flows naturally once we label some side lengths with variable names and start writing down the equations that come from the given information. Wait – that should be one of our standard problem solving techniques – label the diagram with variables where need be and write down equations that result from the given information!

Here is a second similar problem. See if you can solve this problem using an algebraic approach like we used for Problem 1. Then, see if you can come up with a non-algebraic approach that perhaps sheds a different kind of light on the problem.

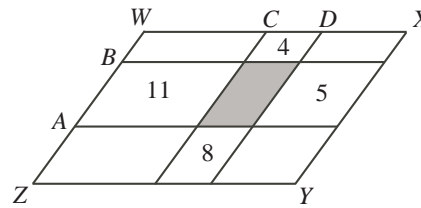


**Problem 2** (2009 UK Junior Mathematics Challenge) The parallelogram  $WXYZ$  shown in the diagram has been divided into nine smaller parallelograms. The perimeters, in cm, of four of the smaller parallelograms are shown. The perimeter of  $WXYZ$  is 21 cm. What is the perimeter of the shaded parallelogram?

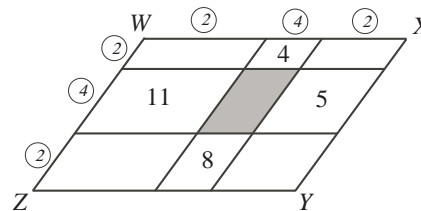


**Solution to Problem 2.** To make things easier to talk about, we label the intermediate points on  $ZW$  and  $WX$  as  $A$ ,  $B$ ,  $C$ , and  $D$ , as shown. Since  $WXYZ$  is a parallelogram and each of its nine sub-regions is a parallelogram, then we can use the fact that opposite sides in a parallelogram are equal in length.

The perimeter of  $WXYZ$  equals  $2WX + 2WZ$ , so  $2WX + 2WZ = 21$ . We'll come back to this information in a minute.



Let's look at each of the four smaller parallelograms whose perimeters we know and relate the sides of these to the segments along  $ZW$  and  $WX$ . For example, the smaller parallelogram with perimeter 11 has 2 sides equal to  $AB$  and 2 sides equal to  $WC$ . We can see this by translating the pieces of the perimeter to the leftmost edge  $WZ$  or the uppermost edge  $WX$ . Try doing this for each of the remaining three smaller parallelograms.



When we have done this for all four smaller parallelograms, we see that each of the segments  $ZA$ ,  $AB$ ,  $BW$ ,  $WC$ ,  $CD$ , and  $DX$  is counted twice, with each of  $AB$  and  $CD$  counted two more times. (The circled numbers in the diagram show how many times a segment is counted.) Therefore, the sum of the perimeters of the four regions (which is  $11 + 4 + 5 + 8$ , or 28) must equal  $2(ZA + AB + BW + WC + CD + DX) + 2(AB + CD)$ .

But the given perimeter of the large parallelogram is 21, and this is equal to  $2(ZW + WX)$ , which actually equals the first term in the previous sum. Therefore,  $21 + 2(AB + CD) = 28$ , or  $2(AB + CD) = 7$ .

Finally, the shaded parallelogram actually has perimeter  $2(AB + CD)$ , by the same translation argument that we used above. Therefore, the perimeter of the shaded parallelogram is 7 cm. ■

For some of us, algebra makes things easier. For some of us, algebra is scary or obscures what's going on. This varies from person to person and problem to problem. Look for different approaches to solve a problem, especially for approaches that give insight into the mechanism of the problem.

## Square Triangles

Peter Hurthig

CRUX Problem 3440 (see [2009 : 233, 236; 2010 : 244-245]) stated:

There are  $N$  coins on a table all of the same size. These  $N$  coins can be arranged in a square and they can also be arranged into an equilateral triangle. Find  $N$ .

We want to find solutions of  $N = S(m) = T(n)$ , where  $T(n) = \frac{1}{2}n(n+1)$  is the  $n^{\text{th}}$  triangular number and  $S(m) = m^2$  is the  $m^{\text{th}}$  square number. The smallest value of  $N$  from Problem 3440 for which a pattern becomes evident is  $N = 36$ , since  $S(6) = T(8) = 36$ . (Actually,  $N = 1$  is a solution since  $S(1) = T(1) = 1$ , but we'll assume that  $N > 1$ .)

We will show the following two facts in a visual way:

- (I) If  $T(x) = 2T(y)$  for some positive integers  $x$  and  $y$ , then the equation  $S(x+y+1) = T(x+2y+1)$  is true.
- (II) If  $S(u) = T(v)$  for some positive integers  $u$  and  $v$ , then the equation  $T(2u+v) = 2T(u+v)$  is true.

These pairs of relations allow us to get another solution to  $S(m) = T(n)$  from a given solution. For example, since  $S(6) = T(8)$ , then (II) gives  $T(20) = 2T(14)$ , from which (I) gives  $S(35) = T(49)$ .

Assume for the moment that we have established facts (I) and (II) and we set  $(m_1, n_1) = (6, 8)$ , which we know is a solution to the problem. In general, if  $S(m_k) = T(n_k)$ , then (II) gives  $T(2m_k + n_k) = 2T(m_k + n_k)$ , from which (I) gives  $S(3m_k + 2n_k + 1) = T(4m_k + 3n_k + 1)$ . Setting

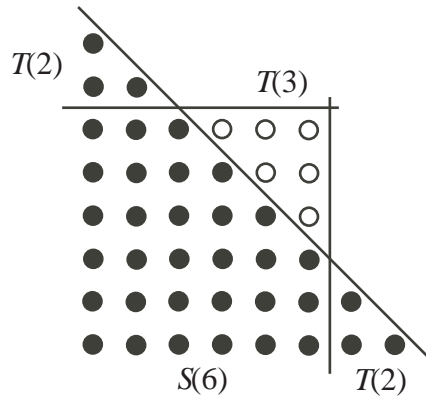
$$(m_{k+1}, n_{k+1}) = (3m_k + 2n_k + 1, 4m_k + 3n_k + 1)$$

allows us to generate an infinite sequence of solutions. The first few solutions generated this way are:

$k$	$m_k$	$n_k$	$N$
1	6	8	36
2	35	49	1225
3	204	288	41 616
4	1189	1681	1 413 721
5	6930	9800	48 024 900

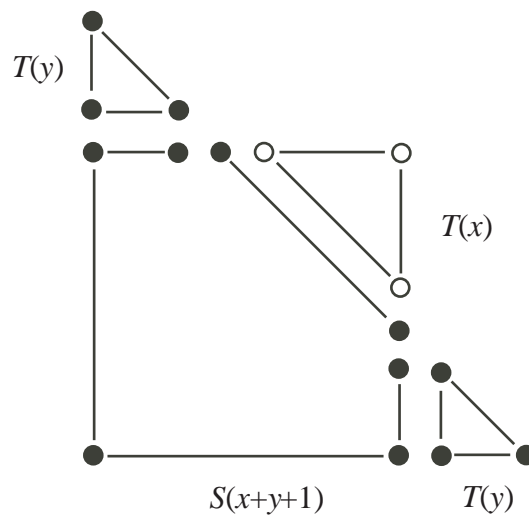
(To answer the question of Problem 3440, if the coins are loonies, then there would be  $N = 36$  on my table, or my table would be very, very large.)

We need to establish properties (I) and (II). We first show that  $T(3) = 2T(2)$  implies that  $S(6) = T(8)$ :



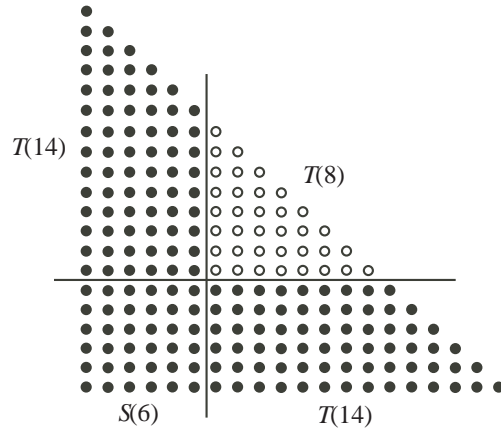
Note that the coins of the two  $T(2)$ -triangles at the end of the  $T(8)$ -triangle fill in the empty spaces of a  $T(3)$ -triangle to complete the  $S(6)$ -square. In other words,  $T(8) = S(6) + 2T(2) - T(3)$  and  $T(3) = 2T(2)$  imply that  $S(6) = T(8)$ .

The following diagram shows that (I) is true in general:



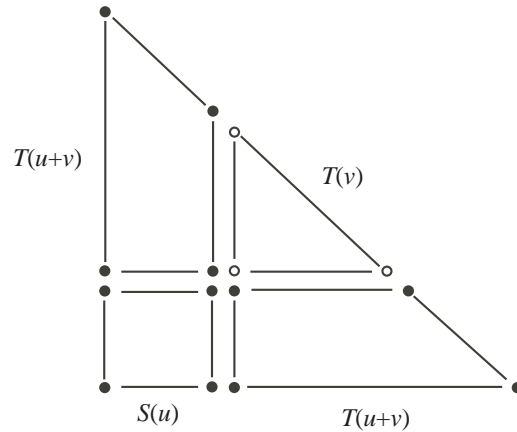
Note that the coins of the two  $T(y)$ -triangles at the end of the  $T(x+2y+1)$ -triangle fill in the empty spaces of a  $T(x)$ -triangle to complete the resulting  $S(x+y+1)$ -square.

Next, we show that  $S(6) = T(8)$  implies that  $T(20) = 2T(14)$ :



The coins of the  $S(6)$ -square at the bottom of the  $T(20)$ -triangle can be stacked on the coins of the  $T(8)$ -triangle to form two overlapping copies of  $T(14)$ -triangle. That is,  $T(20) = 2T(14) - T(8) + S(6)$  and  $S(6) = T(8)$  imply that  $T(20) = 2T(14)$ .

The following diagram shows that (II) is true in general:



Note that the coins of the  $S(u)$ -square at the bottom of the  $T(u + 2v)$ -triangle can be stacked on the coins of the  $T(v)$ -triangle to form two overlapping  $T(u + v)$ -triangles.

These diagrams show that facts (I) and (II) are true. These facts, combined with the algebraic work above, generate the sequence of solutions.

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