Problem of the Month

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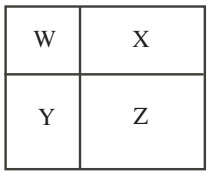
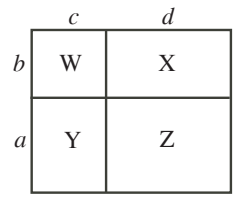
Some problems have an algebraic solution for those who naturally like to convert things to algebra. However, some of us don't naturally immediately think to use algebra, and so it's nice to try to find solutions that are less algebraic. Often this latter type of solution can provide a bit more insight into what is actually going on.



Problem 1 (2010 Fermat Contest) A rectangle is divided into four smaller rectangles, labelled W, X, Y, and Z, as shown. The perimeters of rectangles W, X, and Y are 2, 3, and 5, respectively. What is the perimeter of rectangle Z?

Before seeing this problem, I had seen similar problems involving areas where we are given the areas of three of the sections and asked to find the area of the fourth. I don't remember ever having seen such a problem involving perimeters. Here is an algebraic solution to this problem.

Solution to Problem 1. Label the lengths of the relevant vertical and horizontal segments as a, b, c, and d, as in the diagram.



Rectangle W is b by c, so it has perimeter $2b+2c$. This equals 2. Rectangle X is b by d, so its perimeter is $2b+2d$. This equals 3. Rectangle Y is a by c, so its perimeter is $2a+2c$. This equals 5.

Rectangle Z is a by d, so its perimeter is $2a+2d$. Since $2b+2c=2$ and $2b+2d=3$ and $2a+2c=5$, then

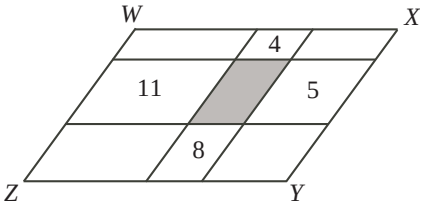
$$\begin{aligned} 2a+2d &= (2a+2b+2c+2d)-(2b+2c) \\ &= (2a+2c)+(2b+2d)-(2b+2c) \\ &= 5+3-2 \;=\; 6\,. \end{aligned}$$

■

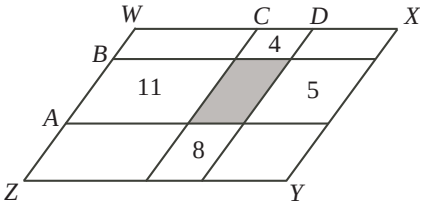
This solution flows naturally once we label some side lengths with variable names and start writing down the equations that come from the given information. Wait – that should be one of our standard problem solving techniques – label the diagram with variables where need be and write down equations that result from the given information!

Here is a second similar problem. See if you can solve this problem using an algebraic approach like we used for Problem 1. Then, see if you can come up with a non-algebraic approach that perhaps sheds a different kind of light on the problem.

Problem 2 (2009 UK Junior Mathematics Challenge) The parallelogram $WXYZ$ shown in the diagram has been divided into nine smaller parallelograms. The perimeters, in cm, of four of the smaller parallelograms are shown. The perimeter of $WXYZ$ is 21 cm. What is the perimeter of the shaded parallelogram?

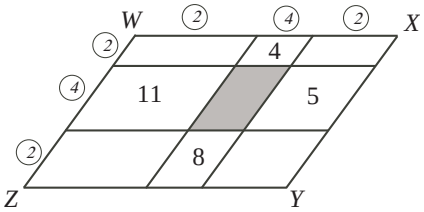


Solution to Problem 2. To make things easier to talk about, we label the intermediate points on ZW and WX as A, B, C, and D, as shown. Since $WXYZ$ is a parallelogram and each of its nine sub-regions is a parallelogram, then we can use the fact that opposite sides in a parallelogram are equal in length.



The perimeter of $WXYZ$ equals $2WX + 2WZ$, so $2WX + 2WZ = 21$. We'll come back to this information in a minute.

Let's look at each of the four smaller parallelograms whose perimeters we know and relate the sides of these to the segments along ZW and WX. For example, the smaller parallelogram with perimeter 11 has 2 sides equal to AB and 2 sides equal to WC. We can see this by translating the pieces of the perimeter to the leftmost edge WZ or the uppermost edge WX. Try doing this for each of the remaining three smaller parallelograms.



When we have done this for all four smaller parallelograms, we see that each of the segments ZA, AB, BW, WC, CD, and DX is counted twice, with each of AB and CD counted two more times. (The circled numbers in the diagram show how many times a segment is counted.) Therefore, the sum of the perimeters of the four regions (which is $11 + 4 + 5 + 8$, or 28) must equal $2(ZA + AB + BW + WC + CD + DX) + 2(AB + CD)$.

But the given perimeter of the large parallelogram is 21, and this is equal to $2(ZW + WX)$, which actually equals the first term in the previous sum. Therefore, $21 + 2(AB + CD) = 28$, or $2(AB + CD) = 7$.

Finally, the shaded parallelogram actually has perimeter $2(AB+CD)$, by the same translation argument that we used above. Therefore, the perimeter of the shaded parallelogram is 7 cm. ■

For some of us, algebra makes things easier. For some of us, algebra is scary or obscures what's going on. This varies from person to person and problem to problem. Look for different approaches to solve a problem, especially for approaches that give insight into the mechanism of the problem.