Square Triangles

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CRUX Problem 3440 (see [2009 : 233, 236; 2010 : 244-245]) stated:

> There are N coins on a table all of the same size. These N coins can be arranged in a square and they can also be arranged into an equilateral triangle. Find N.

We want to find solutions of $N = S(m) = T(n)$, where $T(n) = \frac{1}{2}n(n+1)$ is the n^{th} triangular number and $S(m) = m^2$ is the m^{th} square number. The smallest value of N from Problem 3440 for which a pattern becomes evident is $N = 36$, since $S(6) = T(8) = 36$. (Actually, $N = 1$ is a solution since $S(1) = T(1) = 1$, but we'll assume that $N > 1$.)

We will show the following two facts in a visual way:

(I) If $T(x) = 2T(y)$ for some positive integers x and y, then the equation $S(x+y+1) = T(x+2y+1)$ is true.

(II) If $S(u) = T(v)$ for some positive integers u and v, then the equation $T(2u+v) = 2T(u+v)$ is true.

These pairs of relations allow us to get another solution to $S(m) = T(n)$ from a given solution. For example, since $S(6) = T(8)$, then (II) gives $T(20) = 2T(14)$, from which (I) gives $S(35) = T(49)$.

Assume for the moment that we have established facts (I) and (II) and we set $(m_1, n_1) = (6, 8)$, which we know is a solution to the problem. In general, if $S(m_k) = T(n_k)$, then (II) gives $T(2m_k + n_k) = 2T(m_k + n_k)$, from which (I) gives $S(3m_k + 2n_k + 1) = T(4m_k + 3n_k + 1)$. Setting

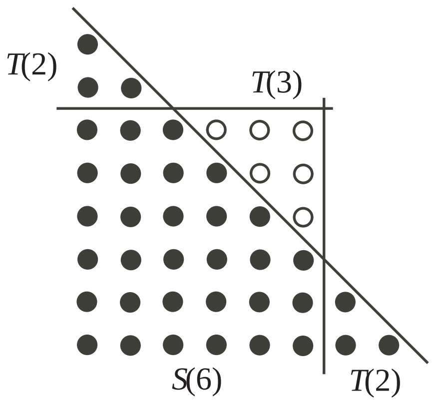
$$(m_{k+1}, n_{k+1}) = (3m_k + 2n_k + 1, 4m_k + 3n_k + 1)$$

allows us to generate an infinite sequence of solutions. The first few solutions generated this way are:

k	m_k	n_k	N
1	6	8	36
2	35	49	1225
3	204	288	41 616
4	1189	1681	1 413 721
5	6930	9800	48 024 900

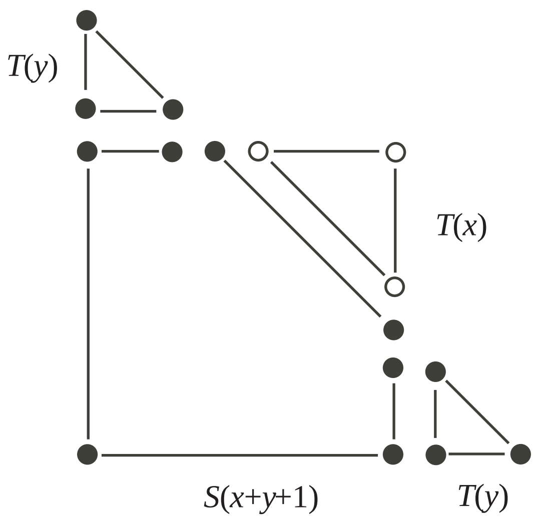
(To answer the question of Problem 3440, if the coins are loonies, then there would be $N = 36$ on my table, or my table would be very, very large.)

We need to establish properties (I) and (II). We first show that $T(3) = 2T(2)$ implies that $S(6) = T(8)$:



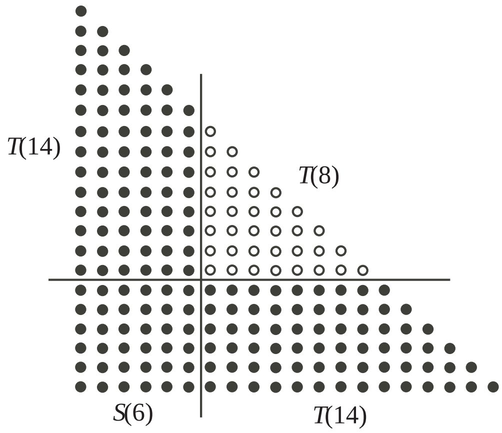
Note that the coins of the two $T(2)$-triangles at the end of the $T(8)$-triangle fill in the empty spaces of a $T(3)$-triangle to complete the $S(6)$-square. In other words, $T(8) = S(6) + 2T(2) - T(3)$ and $T(3) = 2T(2)$ imply that $S(6) = T(8)$.

The following diagram shows that (I) is true in general:



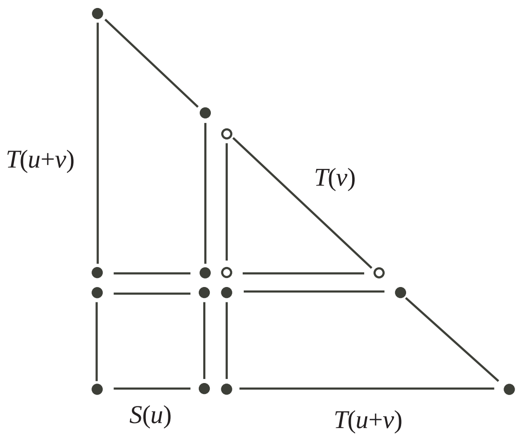
Note that the coins of the two $T(y)$-triangles at the end of the $T(x+2y+1)$-triangle fill in the empty spaces of a $T(x)$-triangle to complete the resulting $S(x + y + 1)$-square.

Next, we show that $S(6) = T(8)$ implies that $T(20) = 2T(14)$:



The coins of the $S(6)$-square at the bottom of the $T(20)$-triangle can be stacked on the coins of the $T(8)$-triangle to form two overlapping copies of $T(14)$-triangle. That is, $T(20) = 2T(14) - T(8) + S(6)$ and $S(6) = T(8)$ imply that $T(20) = 2T(14)$.

The following diagram shows that (II) is true in general:



Note that the coins of the $S(u)$-square at the bottom of the $T(u + 2v)$-triangle can be stacked on the coins of the $T(v)$-triangle to form two overlapping $T(u + v)$-triangles.

These diagrams show that facts (I) and (II) are true. These facts, combined with the algebraic work above, generate the sequence of solutions.

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