

SKOLIAD No. 128

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Please send your solutions to problems in this Skoliad by **1 May 2011**. A copy of *CRUX with Mayhem* will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Mathematics Association of Quebec Contest, Secondary level, 2010. Our thanks go to Marc Bergeron, Cégep de Sainte-Foy, Quebec, for providing us with this contest and for permission to publish it.

Mathematics Association of Quebec Contest, 2010 Secondary level 3 hours allowed

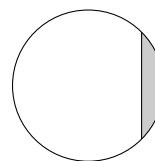
1. An *alphametic* is a small mathematical puzzle consisting of an equation in which the digits have been replaced by letters. The task is to identify the value of each letter in such a way that the equation comes out true. Different letters have different values, different digits are represented by different letters, and no number begins with a zero. For example, the alphametic $PAPA + PAPA = MAMAN$ has the solution $P = 7, A = 5, M = 1, N = 0$, yielding $7575 + 7575 = 15150$.

Find the solution to this “reversing” alphametic:

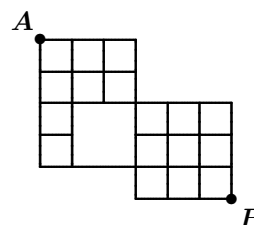
$$\text{NOMBRE} \times \frac{3}{5} = \text{ERBMON}.$$

2. Find all polynomials of the form $p(x) = x^3 + mx + 6$ whose roots are integers.

3. A line is located at $\frac{\sqrt{2}}{2}$ units from the centre of a circle of radius 1, separating it into two parts. What is the area of the smaller part?

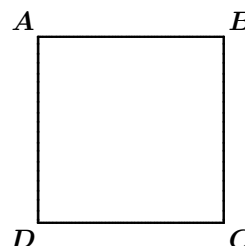


4. The figure shows a map of a city. In how many ways can you travel along the roads of the city from point A to point B if you can only travel east and south (right and down in the figure)?



5. (a) How many zeroes are at the right-hand end of the number $1 \times 2 \times 3 \times \dots \times 52$?
- (b) What is the rightmost nonzero digit of $1 \times 2 \times \dots \times 52$? (For example, the rightmost nonzero digit of $1 \times 2 \times \dots \times 12 = 479\,001\,600$ is 6.)

6. Juliette and Philippe play the following game: At the beginning of the game, each corner of a square is covered with a number of chips. In turn, each player chooses one side of the square and removes as many chips as (s)he wants from the endpoints of that side provided (s)he takes at least one chip. It is not necessary to remove the same number of chips from each endpoint. The player who removes the last chip wins. At the beginning of the game on the square $ABCD$ there are 10 chips on corner A , 11 chips on B , 12 chips on C , and 13 chips on D . If Juliette begins, how should she play?



7. Find all functions $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $F(x) + xF(-x) = 1$ for all real numbers x .

**Concours de l'Association mathématique du Québec,
2010
Ordre secondaire
3 heures a permis**

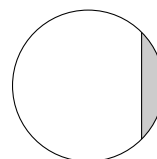
1. Un *alphamétique* est un petit casse-tête mathématique qui consiste en une équation où les chiffres sont remplacés par des lettres. Le résoudre consiste à trouver quelle lettre correspond à quel chiffre pour que l'équation soit vraie. Dans le problème, le même chiffre ne peut être représenté par deux lettres différentes et une lettre représente toujours le même chiffre. Bien entendu, un nombre ne doit jamais commencer par zéro. Par exemple, l'alphamétique $PAPA + PAPA = MAMAN$ a pour solution $P = 7$, $A = 5$, $M = 1$ et $N = 0$. Ainsi, en remplaçant les lettres par les chiffres, on a bien $7575 + 7575 = 15150$.

Trouvez la solution de l'alphamétique "renversant" suivant:

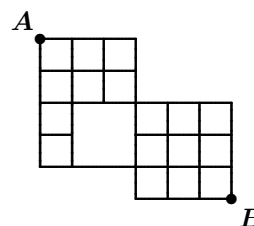
$$\text{NOMBRE} \times \frac{3}{5} = \text{ERBMON}.$$

2. Trouvez tous les polynômes de la forme $p(x) = x^3 + mx + 6$ dont tous les zéros sont des nombres entiers.

3. Une droite est située à $\frac{\sqrt{2}}{2}$ unités du centre d'un cercle de rayon 1, le séparant en deux parties. Quelle est l'aire de la plus petite partie?

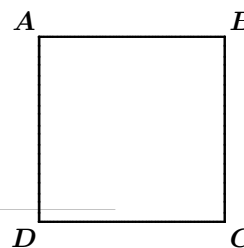


4. La grille suivante représente le plan d'une ville. En partant du point A , combien y a-t-il de chemins (courts) distincts se rendant à B ? (Un chemin court ne va jamais vers le haut de la grille ni vers la gauche.)



5. (a) Combien y a-t-il de zéros à la fin de $1 \times 2 \times 3 \times \dots \times 52$?
 (b) Quel est le dernier chiffre (i.e. le plus à droite) non nul de l'expansion décimale de $1 \times 2 \times 3 \times \dots \times 52$? (Par exemple, le dernier chiffre non nul de $1 \times 2 \times 3 \times \dots \times 12 = 479\,001\,600$ est 6.)

6. Juliette et Philippe jouent au jeu suivant. Au début de la partie, chaque coin d'un carré est recouvert d'un certain nombre de jetons. À tour de rôle, chaque joueur choisit un côté du carré et retire autant de jetons qu'il veut des deux coins qui limitent ce côté, pourvu qu'il en enlève en tout au moins un. Il n'est pas nécessaire de retirer le même nombre de jetons à chacun des coins. Le premier joueur qui se retrouve devant un carré dont tous les coins sont vides a perdu. Au début de la partie, sur le carré $ABCD$, il y a 10 jetons sur le coin A , 11 sur le coin B , 12 sur le coin C et 13 sur le coin D . Si Juliette commence, comment devrait-elle jouer?



7. Quelles sont les fonctions $F : \mathbb{R} \rightarrow \mathbb{R}$ vérifiant $F(x) + xF(-x) = 1$, pour tout x réel?

Next are solutions to the 27th New Brunswick Mathematics Competition, 2009, Grade 9, Part C, given in Skoliad 122 at [2010 : 1–3].

1. If you write all integers from 1 to 100, how many even digits will be written? (When you write the number 42, two even digits are written.)

(A) 50 (B) 71 (C) 80 (D) 89 (E) 91

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

You can simply count the even digits, which is most conveniently done

by setting up a table like this:

1–10	5 even digits,	51–60	6 even digits,
11–20	6 even digits,	61–70	14 even digits,
21–30	14 even digits,	71–80	6 even digits,
31–40	6 even digits,	81–90	14 even digits,
41–50	14 even digits,	91–100	6 even digits,

for a total of 91 even digits.

Also solved by BILLY SUANDITO, Palembang, Indonesia.

Alternatively, you can notice that every second ones digit is even; that contributes 50 digits. The 20's, 40's, 60's, and 80's each contribute ten even tens digits; that is 40 digits. Finally, 100 contributes a single even tens digit. Again, the grand total is 91 even digits.

2. In a farm there are hens (no hump, two legs), camels (two humps, four legs) and dromedaries (one hump, four legs). If the number of legs is four times the number of humps, then the number of hens divided by the number of camels will be?

- (A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) Not enough information

Solution by Billy Suandito, Palembang, Indonesia.

Say the farm has A hens, B camels, and C dromedaries. Then the hens contribute 0 humps and $2A$ legs, the camels contribute $2B$ humps and $4B$ legs, while the dromedaries contribute C humps and $4C$ legs for a total of $2B + C$ humps and $2A + 4B + 4C$ legs. Since the number of legs is given to be four times the number of humps, $2A + 4B + 4C = 4(2B + C) = 8B + 4C$. Thus $2A + 4B = 8B$, so $2A = 4B$, so $A = 2B$. Therefore, $\frac{A}{B} = \frac{2B}{B} = 2$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; and ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration).

3. A cubic box of side 1 m is placed on the floor. A second cubic box of side $\frac{2}{3}$ m is placed on top of the first box so that the centre of the second box is directly above the centre of the first box. A painter paints all of the surface area of the two boxes that can be reached without moving the boxes. What is the total area of surface that is painted?

- (A) $\frac{49}{9}$ m² (B) $\frac{57}{9}$ m² (C) $\frac{61}{9}$ m² (D) $\frac{72}{9}$ m² (E) None of these

Solution by Billy Suandito, Palembang, Indonesia.

The total area of five faces of the large cube is $5 \times 1 \text{ m} \times 1 \text{ m} = 5 \text{ m}^2$. The total area of five faces of the small cube is $5 \times \frac{2}{3} \text{ m} \times \frac{2}{3} \text{ m} = \frac{20}{9} \text{ m}^2$. The small cube hides a $\frac{2}{3} \text{ m} \times \frac{2}{3} \text{ m}$ square of one face of the large cube.

Therefore, the painted area is $5 \text{ m}^2 + \frac{20}{9} \text{ m}^2 - \frac{4}{9} \text{ m}^2 = \frac{61}{9} \text{ m}^2$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration); and one anonymous solver.

4. What is the ones digit of 2^{2009} ?

- (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

You can easily find the ones digit of the first few powers of 2: $2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv 8$, $2^4 \equiv 6$, $2^5 \equiv 2$, $2^6 \equiv 4$, $2^7 \equiv 8$, $2^8 \equiv 6$, etc. mod 10. So the pattern is 2, 4, 8, 6, 2, 4, 8, 6, . . . , which repeats in groups of four. Now, 2009 is not divisible by 4, but $2008 \div 4 = 502$. Therefore, 2^{2008} ends in a 6, so 2^{2009} ends in a 2, because the next number in the pattern is 2.

Also solved by MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; and ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration).

The notation " $2^8 \equiv 6 \pmod{10}$ " (read 2^8 is equivalent to 6 modulo 10) means that 2^8 and 6 leave the same remainder when divided by 10; and, indeed, they both leave the remainder 6.

One ought to prove the pattern, but it follows easily from the fact that the ones digit of a product depends only on the ones digits of the factors.

5. The numbers 1, 2, 3, 4, 5, and 6 are to be arranged in a row. In how many ways can this be done if 2 is always to the left of 4, and 4 is always to the left of 6? (For example 2, 5, 3, 4, 6, 1 is an arrangement with 2 to the left of 4 and 4 to the left of 6.)

- (A) 20 (B) 36 (C) 60 (D) 120 (E) 240

Solution by Billy Suandito, Palembang, Indonesia.

These are the possible arrangements of the numbers 2, 4, and 6:

2	4	6			
2	4		6		
2	4			6	
2	4				6
2		4	6		
2		4		6	
2		4			6
2			4	6	
2			4		6
2				4	6

	2	4	6		
	2	4		6	
	2	4			6
	2		4	6	
	2		4		6
	2			4	6
		2	4	6	
		2	4		6
		2		4	6
			2	4	6

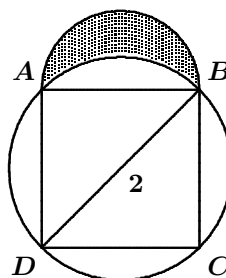
In each of these twenty cases, it remains to place the numbers 1, 3, and 5 in three slots. You have three choices for the placement of 1; for each of these, you have two choices left for the placement of 3; and that leaves just one slot for 5. Hence, in each of the twenty cases, you can place the numbers 1, 3, and 5 in $3 \times 2 \times 1 = 3! = 6$ ways. Therefore, the total number of arrangements is $20 \times 6 = 120$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC.



Our solver found that there are twenty cases by listing them. If you are familiar with counting permutations and combinations, you can also calculate that number: Of the six slots you can choose three for the numbers 2, 4, and 6 in $\binom{6}{3} = \frac{6!}{3!(6-3)!} = 20$ ways. Once you have chosen three slots, the three numbers must go into those slots in the order 2, 4, 6.


6. The square $ABCD$ is inscribed in a circle with diameter BD of length 2. If AB is the diameter of the semicircle on top of the square, what is the area of the shaded region?

- (A) $\frac{4 - \pi}{4}$ (B) $\frac{\pi - 2}{4}$ (C) $\frac{1}{2}$
 (D) 1 (E) Not enough information



Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

By the Pythagorean Theorem, $|AB|^2 + |AD|^2 = |BD|^2$, but $|AD| = |AB|$ and $|BD| = 2$, so $2|AB|^2 = 4$, so $|AB| = \sqrt{2}$. Therefore the area of the square is 2. Clearly the radius of the circle is 1, so the area of the circle is π , whence the area of the shaded region  is $\pi - 2$. Thus the area of a single quarter-segment,  is $\frac{\pi - 2}{4}$.

Since $|AB| = \sqrt{2}$, the radius of the semicircle is $\frac{\sqrt{2}}{2}$, so the area of the semicircle is $\frac{1}{2}\pi \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4}$. It follows that the area of the lune, , is $\frac{\pi}{4} - \frac{\pi - 2}{4} = \frac{1}{2}$.

Also solved by MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; BILLY SUANDITO, Palembang, Indonesia; ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration); and one anonymous solver.

This issue's prize of one copy of **CRUX with MAYHEM** for the best solutions goes to Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC. We look forward to receiving more reader solutions.