SKOLIAD No. 128

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **1 May 2011**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Mathematics Association of Quebec Contest, Secondary level, 2010. Our thanks go to Marc Bergeron, Cégep de Sainte-Foy, Quebec, for providing us with this contest and for permission to publish it.

Mathematics Association of Quebec Contest, 2010
Secondary level
3 hours allowed

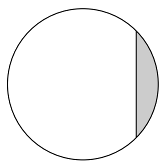
1. An *alphametic* is a small mathematical puzzle consisting of an equation in which the digits have been replaced by letters. The task is to identify the value of each letter in such a way that the equation comes out true. Different letters have different values, different digits are represented by different letters, and no number begins with a zero. For example, the alphametic PAPA + PAPA = MAMAN has the solution P $= 7$, A $= 5$, M $= 1$, and N $= 0$, yielding $7575 + 7575 = 15150$.

Find the solution to this "reversing" alphametic:

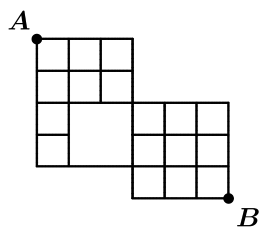
$$\text{NOMBRE} \times \frac{3}{5} = \text{ERBMON}\,.$$

2. Find all polynomials of the form $p(x) = x^3 + mx + 6$ whose roots are integers.

3. A line is located at $\frac{\sqrt{2}}{2}$ units from the centre of a circle of radius 1, separating it into two parts. What is the area of the smaller part?



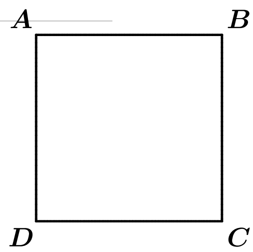
4. The figure shows a map of a city. In how many ways can you travel along the roads of the city from point A to point B if you can only travel east and south (right and down in the figure)?



5. (a) How many zeroes are at the right-hand end of the number $1 \times 2 \times 3 \times \cdots \times 52$?

(b) What is the rightmost nonzero digit of $1 \times 2 \times \cdots \times 52$? (For example, the rightmost nonzero digit of $1 \times 2 \times \cdots \times 12 = 479\,001\,600$ is 6.)

6. Juliette and Philippe play the following game: At the beginning of the game, each corner of a square is covered with a number of chips. In turn, each player chooses one side of the square and removes as many chips as (s)he wants from the endpoints of that side provided (s)he takes at least one chip. It is not necessary to remove the same number of chips from each endpoint. The player who removes the last chip wins. At the beginning of the game on the square $ABCD$ there are 10 chips on corner A, 11 chips on B, 12 chips on C, and 13 chips on D. If Juliette begins, how should she play?



7. Find all functions $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) + xF(-x) = 1$ for all real numbers x.

Concours de l'Association mathématique du Québec, 2010
Ordre secondaire
3 heures a permis

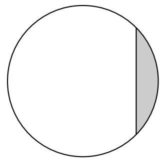
1. Un *alphamétique* est un petit casse-tête mathématique qui consiste en une équation où les chiffres sont remplacés par des lettres. Le résoudre consiste à trouver quelle lettre correspond à quel chiffre pour que l'équation soit vraie. Dans le problème, le même chiffre ne peut être représenté par deux lettres différentes et une lettre représente toujours le même chiffre. Bien entendu, un nombre ne doit jamais commencer par zéro. Par exemple, l'alphamétique PAPA + PAPA = MAMAN a pour solution P = 7, A = 5, M = 1 et N = 0. Ainsi, en remplaçant les lettres par les chiffres, on a bien $7575 + 7575 = 15150$.

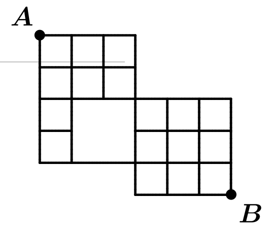
Trouvez la solution de l'alphamétique "renversant" suivant:

$$\text{NOMBRE} \times \frac{3}{5} = \text{ERBMON}\,.$$

2. Trouvez tous les polynômes de la forme $p(x) = x^3 + mx + 6$ dont tous les zéros sont des nombres entiers.

3. Une droite est située à $\frac{\sqrt{2}}{2}$ unités du centre d'un cercle de rayon 1, le séparant en deux parties. Quelle est l'aire de la plus petite partie?



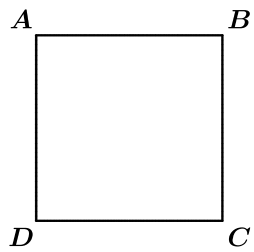


4. La grille suivante représente le plan d'une ville. En partant du point A, combien y a-t-il de chemins (courts) distincts se rendant à B? (Un chemin court ne va jamais vers le haut de la grille ni vers la gauche.)

5. (a) Combien y a-t-il de zéros à la fin de $1 \times 2 \times 3 \times \cdots \times 52$?

(b) Quel est le dernier chiffre (i.e. le plus à droite) non nul de l'expansion décimale de $1 \times 2 \times 3 \times \cdots \times 52$? (Par exemple, le dernier chiffre non nul de $1 \times 2 \times 3 \times \cdots \times 12 = 479\,001\,600$ est 6.)

6. Juliette et Philippe jouent au jeu suivant. Au début de la partie, chaque coin d'un carré est recouvert d'un certain nombre de jetons. À tour de rôle, chaque joueur choisit un côté du carré et retire autant de jetons qu'il veut des deux coins qui limitent ce côté, pourvu qu'il en enlève en tout au moins un. Il n'est pas nécessaire de retirer le même nombre de jetons à chacun des coins. Le premier joueur qui se retrouve devant un carré dont tous les coins sont vides a perdu. Au début de la partie, sur le carré $ABCD$, il y a 10 jetons sur le coin A, 11 sur le coin B, 12 sur le coin C et 13 sur le coin D. Si Juliette commence, comment devrait-elle jouer?



7. Quelles sont les fonctions $F : \mathbb{R} \to \mathbb{R}$ vérifiant $F(x) + xF(-x) = 1$, pour tout x réel?

Next are solutions to the 27$^{\text{th}}$ New Brunswick Mathematics Competition, 2009, Grade 9, Part C, given in Skoliad 122 at [2010 : 1–3].

1. If you write all integers from 1 to 100, how many even digits will be written? (When you write the number 42, two even digits are written.)

(A) 50 (B) 71 (C) 80 (D) 89 (E) 91

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

You can simply count the even digits, which is most conveniently done by setting up a table like this:

1–10	5 even digits,	51–60	6 even digits,
11–20	6 even digits,	61–70	14 even digits,
21–30	14 even digits,	71–80	6 even digits,
31–40	6 even digits,	81–90	14 even digits,
41–50	14 even digits,	91–100	6 even digits,

for a total of 91 even digits.

Also solved by BILLY SUANDITO, Palembang, Indonesia.

Alternatively, you can notice that every second ones digit is even; that contributes 50 digits. The 20's, 40's, 60's, and 80's each contribute ten even tens digits; that is 40 digits. Finally, 100 contributes a single even tens digit. Again, the grand total is 91 even digits.

2. In a farm there are hens (no hump, two legs), camels (two humps, four legs) and dromedaries (one hump, four legs). If the number of legs is four times the number of humps, then the number of hens divided by the number of camels will be?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) Not enough information

Solution by Billy Suandito, Palembang, Indonesia.

Say the farm has A hens, B camels, and C dromedaries. Then the hens contribute 0 humps and $2A$ legs, the camels contribute $2B$ humps and $4B$ legs, while the dromedaries contribute C humps and $4C$ legs for a total of $2B + C$ humps and $2A + 4B + 4C$ legs. Since the number of legs is given to be four times the number of humps, $2A+4B+4C = 4(2B+C) = 8B+4C$. Thus $2A + 4B = 8B$, so $2A = 4B$, so $A = 2B$. Therefore, $\frac{A}{B} = \frac{2B}{B} = 2$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; and ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration).

3. A cubic box of side 1 m is placed on the floor. A second cubic box of side $\frac{2}{3}$ m is placed on top of the first box so that the centre of the second box is directly above the centre of the first box. A painter paints all of the surface area of the two boxes that can be reached without moving the boxes. What is the total area of surface that is painted?

(A) $\frac{49}{9}$ m^2 (B) $\frac{57}{9}$ m^2 (C) $\frac{61}{9}$ m^2 (D) $\frac{72}{9}$ m^2 (E) None of these

Solution by Billy Suandito, Palembang, Indonesia.

The total area of five faces of the large cube is $5 \times 1\,\mathrm{m} \times 1\,\mathrm{m} = 5\,\mathrm{m}^2$. The total area of five faces of the small cube is $5 \times \frac{2}{3}\,\mathrm{m} \times \frac{2}{3}\,\mathrm{m} = \frac{20}{9}\,\mathrm{m}^2$. The small cube hides a $\frac{2}{3}\,\mathrm{m} \times \frac{2}{3}\,\mathrm{m}$ square of one face of the large cube.

Therefore, the painted area is $5\,\mathrm{m}^2 + \frac{20}{9}\,\mathrm{m}^2 - \frac{4}{9}\,\mathrm{m}^2 = \frac{61}{9}\,\mathrm{m}^2$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration); and one anonymous solver.

4. What is the ones digit of 2^{2009}?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

You can easily find the ones digit of the first few powers of 2: $2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv 8$, $2^4 \equiv 6$, $2^5 \equiv 2$, $2^6 \equiv 4$, $2^7 \equiv 8$, $2^8 \equiv 6$, etc. mod 10. So the pattern is $2, 4, 8, 6, 2, 4, 8, 6, \ldots$, which repeats in groups of four. Now, 2009 is not divisible by 4, but $2008 \div 4 = 502$. Therefore, 2^{2008} ends in a 6, so 2^{2009} ends in a 2, because the next number in the pattern is 2.

Also solved by MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; and ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration).

The notation "$2^8 \equiv 6 \pmod{10}$" (read 2^8 is equivalent to 6 modulo 10) means that 2^8 and 6 leave the same remainder when divided by 10; and, indeed, they both leave the remainder 6.

One ought to prove *the pattern, but it follows easily from the fact that the ones digit of a product depends only on the ones digits of the factors.*

5. The numbers 1, 2, 3, 4, 5, and 6 are to be arranged in a row. In how many ways can this be done if 2 is always to the left of 4, and 4 is always to the left of 6? (For example $2, 5, 3, 4, 6, 1$ is an arrangement with 2 to the left of 4 and 4 to the left of 6.)

(A) 20 (B) 36 (C) 60 (D) 120 (E) 240

Solution by Billy Suandito, Palembang, Indonesia.

These are the possible arrangements of the numbers 2, 4, and 6:

2	4	6			
2	4		6		
2	4			6	
2	4				6
2		4	6		
2		4		6	
2		4			6
2			4	6	
2			4		6
2				4	6

	2	4	6		
	2	4		6	
	2	4			6
	2		4	6	
	2		4		6
	2			4	6
		2	4	6	
		2	4		6
		2		4	6
			2	4	6

In each of these twenty cases, it remains to place the numbers 1, 3, and 5 in three slots. You have three choices for the placement of 1; for each of these, you have two choices left for the placement of 3; and that leaves just one slot for 5. Hence, in each of the twenty cases, you can place the numbers 1, 3, and 5 in $3 \times 2 \times 1 = 3! = 6$ ways. Therefore, the total number of arrangements is $20 \times 6 = 120$.

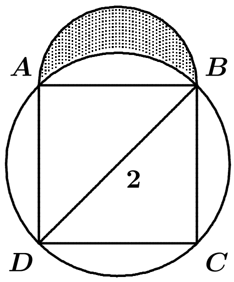
Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC.

Our solver found that there are twenty cases by listing them. If you are familiar with counting permutations and combinations, you can also calculate that number: Of the six slots you can choose three for the numbers 2, 4, and 6 in $\binom{6}{3} = \frac{6!}{3!(6-3)!} = 20$ ways. Once you have chosen three slots, the three numbers must go into those slots in the order 2, 4, 6.

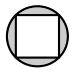
6. The square $ABCD$ is inscribed in a circle with diameter BD of length 2. If AB is the diameter of the semicircle on top of the square, what is the area of the shaded region?

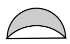
(A) $\frac{4-\pi}{4}$ (B) $\frac{\pi-2}{4}$ (C) $\frac{1}{2}$

(D) 1 (E) Not enough information



Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

By the Pythagorean Theorem, $|AB|^2 + |AD|^2 = |BD|^2$, but $|AD| = |AB|$ and $|BD| = 2$, so $2|AB|^2 = 4$, so $|AB| = \sqrt{2}$. Therefore the area of the square is 2. Clearly the radius of the circle is 1, so the area of the circle is π, whence the area of the shaded region  is $\pi - 2$. Thus the area of a single quarter-segment, , is $\frac{\pi-2}{4}$.

Since $|AB| = \sqrt{2}$, the radius of the semicircle is $\frac{\sqrt{2}}{2}$, so the area of the semicircle is $\frac{1}{2}\pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4}$. It follows that the area of the lune, , is $\frac{\pi}{4} - \frac{\pi-2}{4} = \frac{1}{2}$.

Also solved by MONICA HSIEH, student, Burnaby North Secondary School, Burnaby, BC; BILLY SUANDITO, Palembang, Indonesia; ELLEN CHEN, WEN-TING FAN, VICKY LIAO, JUSTIN MIAO (all students at Burnaby North Secondary School, Burnaby, BC), and LISA WANG, student, Port Moody Secondary School, Port Moody, BC (in collaboration); and one anonymous solver.

This issue's prize of one copy of ***CRUX with MAYHEM*** for the best solutions goes to Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC. We look forward to receiving more reader solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff member is Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON). The editor thanks Rachael Verbruggen, University of Waterloo, for her assistance with this month's solutions.

Mayhem Problems

*Please send your solutions to the problems in this edition by **15 February 2011**. Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M457. *Proposed by the Mayhem Staff.*

Suppose that A is a digit between 0 and 9, inclusive, and that the tens digit of the product of $2A7$ and 39 is 9. Determine the digit A.

M458. *Proposed by the Mayhem Staff.*

Convex quadrilateral $ABCD$ has $AB = AD = 10$ and $BC = CD$. Also, AC is perpendicular to BD, with AC and BD intersecting at P. If $BP = 8$ and $CD = CP + 2$, determine the area of quadrilateral $ABCD$.

M459. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine whether or not it is possible to create a collection of ten distinct subsets of $S = \{1, 2, 3, 4, 5, 6\}$ so that each subset contains three elements, each element of S appears in five subsets, and each pair of elements from S appears in two subsets.

M460. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let a and b be positive real numbers. Define $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, and $K = \sqrt{\frac{a^2+b^2}{2}}$. Prove that (a) $G^2 + K^2 = 2A^2$, (b) $A^2 \geq KG$, (c) $G + K \leq 2A$, and (d) $G^4 + K^4 \geq 2A^4$.

M461. *Proposed by Landelino Arboniés, Colegio Marcelino Champagnat, Santo Domingo, Dominican Republic.*

A *Champagnat* number is equal to the sum of all the digits in a set of consecutive positive integers, one of which is the number itself. Thus, 42 is a Champagnat number, since 42 is the sum of all of the digits of 39, 40, 41, 42, 43, 44. Prove that there exist infinitely many Champagnat numbers.

M462. *Proposed by Alex Song, Detroit Country Day School, Detroit, MI, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x and let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. For example, $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$, $\lfloor -1.4 \rfloor = -2$, and $\lceil -1.4 \rceil = -1$. Determine all real numbers x for which $\lfloor x \rfloor \lceil x \rceil = x^2$.

. . . .

M457. *Proposé par l'Équipe de Mayhem.*

On suppose que A est un chiffre entre 0 et 9 inclusivement, et que le chiffre des dizaines du produit $2A7$ et 39 est 9. Trouver A.

M458. *Proposé par l'Équipe de Mayhem.*

Soit $ABCD$ un quadrilatère convexe avec $AB = AD = 10$ et $BC = CD$. De plus, soit AC perpendiculaire à BD et P leur point d'intersection. Si $BP = 8$ et $CD = CP + 2$, trouver l'aire du quadrilatère $ABCD$.

M459. *Proposé par Neven Jurič, Zagreb, Croatie.*

Déterminer si oui ou non, il est possible de créer une collection de dix sous-ensembles distincts de $S = \{1, 2, 3, 4, 5, 6\}$ de sorte que chaque sous-ensemble contienne trois éléments, que chaque élément de S apparaisse dans cinq sous-ensembles, et que chaque paire d'éléments de S apparaisse dans deux sous-ensembles.

M460. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit a et b deux nombres réels positifs. On définit $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ et $K = \sqrt{\frac{a^2+b^2}{2}}$. Montrer que (a) $G^2 + K^2 = 2A^2$, (b) $A^2 \geq KG$, (c) $G + K \leq 2A$, et (d) $G^4 + K^4 \geq 2A^4$.

M461. *Proposé par Landelino Arboniés, Colegio Marcelino Champagnat, Santo Domingo, République dominicaine.*

Un nombre de Champagnat n est défini comme la somme de tous les chiffres d'une suite d'entiers consécutifs comprenant n. Ainsi, 42 est un nombre de Champagnat puisqu'il est dans la suite 39, 40, 41, 42, 43, 44. Montrer qu'il existe une infinité de nombres de Champagnat.

M462. *Proposé par Alex Song, Detroit Country Day School, Detroit, MI, É-U et Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

On désigne par $\lfloor x \rfloor$ le plus grand entier n'excédant pas x et par $\lceil x \rceil$ le plus petit entier plus grand ou égal à x. Par exemple, $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$, $\lfloor -1.4 \rfloor = -2$, et $\lceil -1.4 \rceil = -1$. Déterminer tous les nombres réels x pour lesquels $\lfloor x \rfloor \lceil x \rceil = x^2$.

Mayhem Solutions

M420. *Proposed by the Mayhem Staff.*

Riley is a poor starving university student, but is mathematically astute. He notices that five suppers in residence cost the same as seven lunches. After one week of skipping supper most nights, he notices that five lunches and one supper cost $\$48$ in total. How much do 16 suppers cost?

Solution by Oscar Xia, student, St. George's School, Vancouver, BC.

Let one lunch cost x dollars and one supper cost y dollars.

Since five suppers cost the same as seven lunches, then $5y = 7x$. Since five suppers and one lunch cost $\$48$, then $5x + y = 48$.

From the first equation, $x = \frac{5}{7}y$ and so we obtain $5\left(\frac{5}{7}y\right) + y = 48$, or $\frac{32}{7}y = 48$, or $y = \frac{21}{2}$.

Therefore, 16 suppers cost $16\left(\frac{21}{2}\right) = 168$ dollars.

Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; LEWIS HUGHES, Auburn University, Montgomery, AL, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M421. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let $\lfloor x \rfloor$ be the greatest integer less than or equal to the real number x. Determine all real numbers x such that

$$\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{3}{x} \right\rfloor = 4\,.$$

Solution by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA.

Since x cannot be 0, then we let $y = \frac{1}{x}$. Therefore, we are looking for all real nonzero solutions to $\lfloor y \rfloor + \lfloor 3y \rfloor = 4$. Note that y cannot be negative, because if it were, then the left side of the equation would be negative. Therefore, y is positive.

If $y < 1$, then $3y < 3$ and so $\lfloor y \rfloor + \lfloor 3y \rfloor < y + 3y < 1 + 3 = 4$. Therefore, $y \geq 1$.

If $y \geq \frac{4}{3}$, then $3y \geq 4$. In this case, $\lfloor y \rfloor \geq 1$ and $\lfloor 3y \rfloor \geq 4$, so $\lfloor y \rfloor + \lfloor 3y \rfloor \geq 5$. Therefore, $y < \frac{4}{3}$.

Combining what we know, we have $1 \leq y < \frac{4}{3}$. For these y, we have $\lfloor y \rfloor = 1$. Also, we have $3 \leq 3y < 4$ and so $\lfloor 3y \rfloor = 3$. Therefore, for all of these y, we have $\lfloor y \rfloor + \lfloor 3y \rfloor = 4$.

Therefore, the values of y that satisfy the equation are $1 \leq y < \frac{4}{3}$, and so the values of x that satisfy the original equation are $\frac{3}{4} < x \leq 1$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; OSCAR XIA, student, St. George's School, Vancouver, BC; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. One incorrect solution and two incomplete solutions were submitted.

M422. *Proposed by Adnan Arapovic, student, University of Waterloo, Waterloo, ON.*

Prove that

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Since $\binom{k+1}{2} = \frac{k(k+1)}{2}$, then the given equation is equivalent to

$$\sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{3}.$$

The total number of 3-element subsets of $\{1, 2, 3, \ldots, n+1, n+2\}$ is equal to $\binom{n+2}{3}$.

These subsets can also be counted in the following way. First, choose the largest element, M, in the subset. Note that M can take any value from 3 to $n+2$. Next, choose the remaining two elements from among the numbers 1 to $M-1$. Let $M = k+2$, where $k = 1, 2, \ldots, n$. For a fixed value of k, there are $\binom{k+1}{2}$ ways of choosing the remaining two elements. Summing over all possible values of k, we see that the total number of 3-element subsets of $\{1, 2, 3, \ldots, n+1, n+2\}$ is also equal to $\sum_{k=1}^{n} \binom{k+1}{2}$.

Therefore, $\sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{3}$.

Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; KONSTANTINOS AL. NAKOS, Agrinio, Greece; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON (second and third solutions); JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; OSCAR XIA, student, St. George's School, Vancouver, BC; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Three incomplete solutions were submitted.

M423. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The tens digit of a perfect square S is three greater than the ones digit of S. Determine all possible remainders when S is divided by 100.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Every perfect square has its units digit in the set $\{0, 1, 4, 5, 6, 9\}$. (This can be seen by squaring the integers from 0 to 9 and observing that the pattern of units digits continues for larger squares.)

Since the tens digit is to be 3 greater than the units digit, then units digit cannot be greater than 6, so the only possible remainders when S is divided by 100 (that is, the possible pairs of last two digits of S) are 30, 41, 74, 85, and 96. That is, S must have one of the following forms

$$100k + 30\,,\quad 100k + 41\,,\quad 100k + 74\,,\quad 100k + 85\,,\quad 100k + 96$$

for some integer k.

Note that even perfect squares must be divisible by 4, because an even perfect square is the square of an even integer, say $2n$ for some integer n, so the square is $4n^2$. Any integer of the form $100k + 30$ or $100k + 74$ is not divisible by 4, so cannot be a perfect square. Thus, S cannot end in 30 or 74.

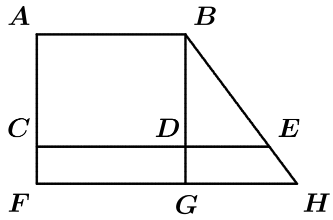
Also, S cannot end in 85. If S did end in 85, then its square root would end in 5, so S would be of the form $(10a + 5)^2$ for some integer a. In this case, $S = 100a^2 + 100a + 25$, which ends in 25. So S cannot end in 85.

Therefore, the only possibilities for the last two digits of S are 41 and 96. Each of them is possible since $21^2 = 441$ and $14^2 = 196$.

Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain. One incomplete solution and one incorrect solution were submitted.

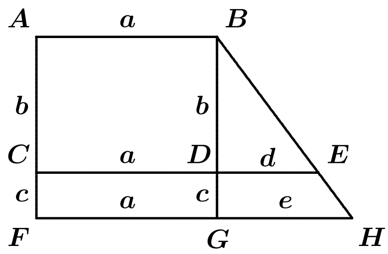
M424. *Proposed by Margo Kondratieva, Memorial University of Newfoundland, St. John's, NL.*

In the diagram, line segments AB, CDE, and FGH are parallel. Also, line segments ACF and BDG are perpendicular to AB. Suppose that the area of rectangle $ABDC$ is x, the area of rectangle $CDGF$ is y, and the area of $\triangle BDE$ is z. Determine the area of $DEHG$ in terms of x, y, and z.



Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

Since line segments AB, CDE, and FGH are parallel, and line segments ACF and BDG are perpendicular to AB, then AB, CD, and FGH are each perpendicular to both ACF and BDG. Therefore, $ABDC$ and $CDGF$ are rectangles, and $\triangle BDE$ is similar to $\triangle BGH$, since they share a common angle at B.



Suppose that $AB = CD = FG = a$, $AC = BD = b$, $CF = DG = c$, $DE = d$, and $GH = e$.

Since $\triangle BDE$ is similar to $\triangle BGH$, we then have $\dfrac{DE}{GH} = \dfrac{BD}{BG}$, and so $\dfrac{d}{e} = \dfrac{b}{b+c}$, which gives $e = \dfrac{d(b+c)}{b}$.

Since the area of $ABDC$ is x, then $ab = x$. Since the area of $CDGF$ is y, then $ac = y$. Since the area of $\triangle BDE$ is z, then $bd = 2z$.

From the first two equations, $\dfrac{x}{y} = \dfrac{ab}{ac} = \dfrac{b}{c}$. From the second and third equations, $abcd = 2yz$, which gives $cd = \dfrac{2yz}{ab} = \dfrac{2yz}{x}$.

Since $DEHG$ is a trapezoid, then

$$\begin{aligned}
&\text{Area of } DEHG \\
&= \frac{1}{2}(d+e)c = \frac{1}{2}cd + \frac{1}{2}ce = \frac{1}{2}cd + \frac{1}{2}c \cdot \frac{d(b+c)}{b} \\
&= \frac{1}{2}cd + \frac{1}{2}cd + \frac{1}{2} \cdot \frac{dc^2}{b} = cd + \frac{c^2 d}{b} = cd + \frac{1}{2}cd \cdot \frac{c}{b} \\
&= \frac{2yz}{x} + \frac{1}{2} \cdot \frac{2yz}{x} \cdot \frac{y}{x} = \frac{2xyz + y^2 z}{x^2}.
\end{aligned}$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; G.C. GREUBEL, Newport News, VA, USA; SZÉP GYUSZI, Dimitrie Leonida Technological High School, Petrosani, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES

"Abastos", Valencia, Spain; GILI RUSAK, student, Shaker Junior High School, Loudanville, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. One incorrect solution was submitted.

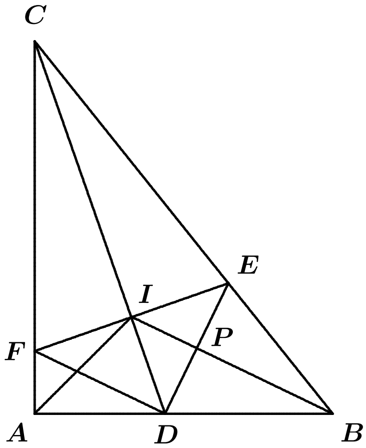
M425. *Proposed by Titu Zvonaru, Comănești, Romania.*

In $\triangle ABC$, $\angle BAC = 90°$ and I is the incentre. The interior bisector of angle C meets AB at D. The line segment through D perpendicular to BI intersects BC at E. The line segment through D parallel to BI meets AC at F. Prove that E, I, and F are collinear.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Since $\triangle ABC$ is right-angled at A, then $\angle ABC + \angle ACB = 90°$.

Consider $\triangle CIB$. We see that $\angle DIB$ is an external angle of this triangle, so $\angle DIB = \angle IBC + \angle ICB$. Since CI and BI bisect $\angle ACB$ and $\angle ABC$, respectively, then $\angle IBC = \frac{1}{2}\angle ABC$ and $\angle ICB = \frac{1}{2}\angle ACB$. Therefore, $\angle DIB = \frac{1}{2}(\angle ABC + \angle ACB) = \frac{1}{2}(90°) = 45°$.



Since DF is parallel to BI, then $\angle FDI = \angle DIB = 45°$. Since AI bisects $\angle BAC$, then $\angle FAI = \angle DAI = 45°$. Thus, FI subtends equal angles at A and at D. Therefore, quadrilateral $FADI$ is cyclic.

In addition, chord DI in cyclic quadrilateral subtends both $\angle DAI$ and $\angle DFI$. Thus, $\angle DFI = \angle DAI = 45°$. We then see that in $\triangle FID$, there are two $45°$ angles, so $\angle FID = 90°$.

Suppose that BI and DE intersect at P. Since BI and DE are perpendicular and $\angle EBP = \angle DBP$ (because BI is an angle bisector) and BP is a common side in $\triangle EBP$ and $\angle DBP$, then these two triangles are congruent. Therefore, $DP = EP$.

This tells us that $\triangle IDP$ and $\triangle IEP$ are congruent, since the triangles have the side IP in common, $DP = EP$, and $\angle IPD = \angle IPE = 90°$. Therefore, $\angle DIP = \angle EIP$.

Thus, $\angle DIE = 2\angle DIP = 2\angle DIB = 90°$.

Finally, $\angle FIE = \angle FID + \angle DIE = 90° + 90° = 180°$. This tells us that E, I, and F are collinear, as required.

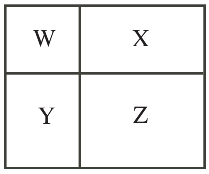
Also solved by MATTHEW BABBITT, home-schooled student, Fort Edward, NY, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Three incorrect solutions were submitted.

Problem of the Month

Ian VanderBurgh

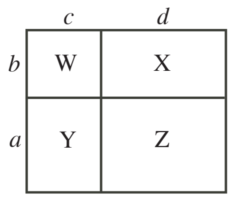
Some problems have an algebraic solution for those who naturally like to convert things to algebra. However, some of us don't naturally immediately think to use algebra, and so it's nice to try to find solutions that are less algebraic. Often this latter type of solution can provide a bit more insight into what is actually going on.

Problem 1 (2010 Fermat Contest) A rectangle is divided into four smaller rectangles, labelled W, X, Y, and Z, as shown. The perimeters of rectangles W, X, and Y are 2, 3, and 5, respectively. What is the perimeter of rectangle Z?



Before seeing this problem, I had seen similar problems involving areas where we are given the areas of three of the sections and asked to find the area of the fourth. I don't remember ever having seen such a problem involving perimeters. Here is an algebraic solution to this problem.

Solution to Problem 1. Label the lengths of the relevant vertical and horizontal segments as a, b, c, and d, as in the diagram.



Rectangle W is b by c, so it has perimeter $2b + 2c$. This equals 2. Rectangle X is b by d, so its perimeter is $2b + 2d$. This equals 3. Rectangle Y is a by c, so its perimeter is $2a + 2c$. This equals 5.

Rectangle Z is a by d, so its perimeter is $2a + 2d$. Since $2b + 2c = 2$ and $2b + 2d = 3$ and $2a + 2c = 5$, then

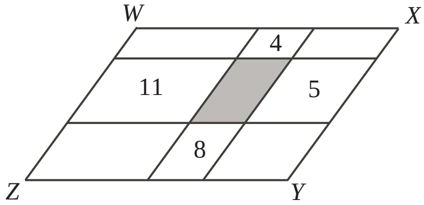
$$\begin{aligned} 2a + 2d &= (2a + 2b + 2c + 2d) - (2b + 2c) \\ &= (2a + 2c) + (2b + 2d) - (2b + 2c) \\ &= 5 + 3 - 2 \;=\; 6\,. \end{aligned}$$

∎

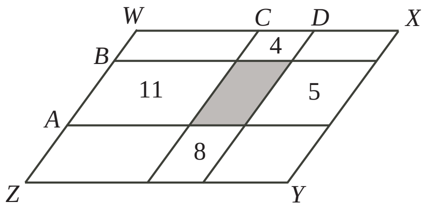
This solution flows naturally once we label some side lengths with variable names and start writing down the equations that come from the given information. Wait – that should be one of our standard problem solving techniques – label the diagram with variables where need be and write down equations that result from the given information!

Here is a second similar problem. See if you can solve this problem using an algebraic approach like we used for Problem 1. Then, see if you can come up with a non-algebraic approach that perhaps sheds a different kind of light on the problem.

Problem 2 (2009 UK Junior Mathematics Challenge) The parallelogram $WXYZ$ shown in the diagram has been divided into nine smaller parallelograms. The perimeters, in cm, of four of the smaller parallelograms are shown. The perimeter of $WXYZ$ is 21 cm. What is the perimeter of the shaded parallelogram?

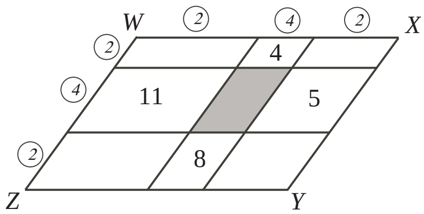


Solution to Problem 2. To make things easier to talk about, we label the intermediate points on ZW and WX as A, B, C, and D, as shown. Since $WXYZ$ is a parallelogram and each of its nine sub-regions is a parallelogram, then we can use the fact that opposite sides in a parallelogram are equal in length.



The perimeter of $WXYZ$ equals $2WX + 2WZ$, so $2WX + 2WZ = 21$. We'll come back to this information in a minute.

Let's look at each of the four smaller parallelograms whose perimeters we know and relate the sides of these to the segments along ZW and WX. For example, the smaller parallelogram with perimeter 11 has 2 sides equal to AB and 2 sides equal to WC. We can see this by translating the pieces of the perimeter to the leftmost edge WZ or the uppermost edge WX. Try doing this for each of the remaining three smaller parallelograms.



When we have done this for all four smaller parallelograms, we see that each of the segments ZA, AB, BW, WC, CD, and DX is counted twice, with each of AB and CD counted two more times. (The circled numbers in the diagram show how many times a segment is counted.) Therefore, the sum of the perimeters of the four regions (which is $11 + 4 + 5 + 8$, or 28) must equal $2(ZA + AB + BW + WC + CD + DX) + 2(AB + CD)$.

But the given perimeter of the large parallelogram is 21, and this is equal to $2(ZW + WX)$, which actually equals the first term in the previous sum. Therefore, $21 + 2(AB + CD) = 28$, or $2(AB + CD) = 7$.

Finally, the shaded parallelogram actually has perimeter $2(AB+CD)$, by the same translation argument that we used above. Therefore, the perimeter of the shaded parallelogram is 7 cm. ∎

For some of us, algebra makes things easier. For some of us, algebra is scary or obscures what's going on. This varies from person to person and problem to problem. Look for different approaches to solve a problem, especially for approaches that give insight into the mechanism of the problem.

Square Triangles

Peter Hurthig

CRUX Problem 3440 (see [2009 : 233, 236; 2010 : 244-245]) stated:

> There are N coins on a table all of the same size. These N coins can be arranged in a square and they can also be arranged into an equilateral triangle. Find N.

We want to find solutions of $N = S(m) = T(n)$, where $T(n) = \frac{1}{2}n(n+1)$ is the n^{th} triangular number and $S(m) = m^2$ is the m^{th} square number. The smallest value of N from Problem 3440 for which a pattern becomes evident is $N = 36$, since $S(6) = T(8) = 36$. (Actually, $N = 1$ is a solution since $S(1) = T(1) = 1$, but we'll assume that $N > 1$.)

We will show the following two facts in a visual way:

(I) If $T(x) = 2T(y)$ for some positive integers x and y, then the equation $S(x+y+1) = T(x+2y+1)$ is true.

(II) If $S(u) = T(v)$ for some positive integers u and v, then the equation $T(2u+v) = 2T(u+v)$ is true.

These pairs of relations allow us to get another solution to $S(m) = T(n)$ from a given solution. For example, since $S(6) = T(8)$, then (II) gives $T(20) = 2T(14)$, from which (I) gives $S(35) = T(49)$.

Assume for the moment that we have established facts (I) and (II) and we set $(m_1, n_1) = (6, 8)$, which we know is a solution to the problem. In general, if $S(m_k) = T(n_k)$, then (II) gives $T(2m_k + n_k) = 2T(m_k + n_k)$, from which (I) gives $S(3m_k + 2n_k + 1) = T(4m_k + 3n_k + 1)$. Setting

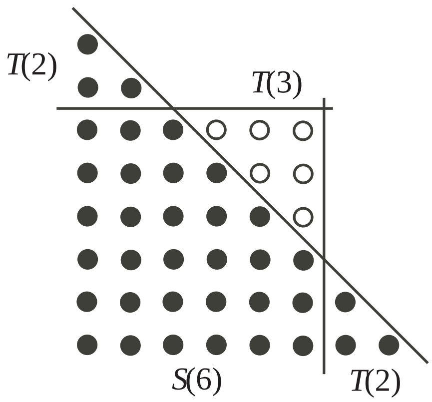
$$(m_{k+1}, n_{k+1}) \;=\; (3m_k + 2n_k + 1, 4m_k + 3n_k + 1)$$

allows us to generate an infinite sequence of solutions. The first few solutions generated this way are:

k	m_k	n_k	N
1	6	8	36
2	35	49	1225
3	204	288	41 616
4	1189	1681	1 413 721
5	6930	9800	48 024 900

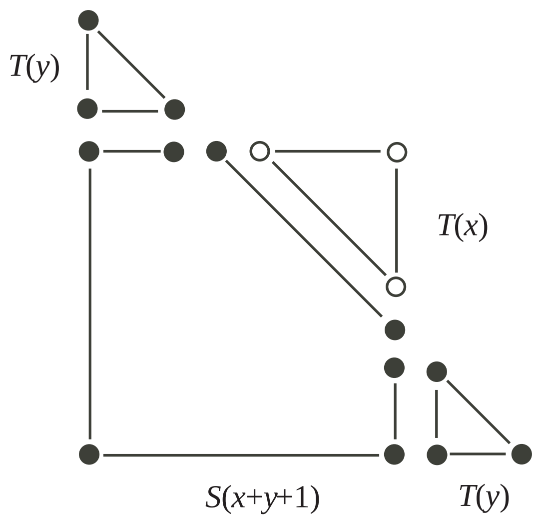
(To answer the question of Problem 3440, if the coins are loonies, then there would be $N = 36$ on my table, or my table would be very, very large.)

We need to establish properties (I) and (II). We first show that $T(3) = 2T(2)$ implies that $S(6) = T(8)$:



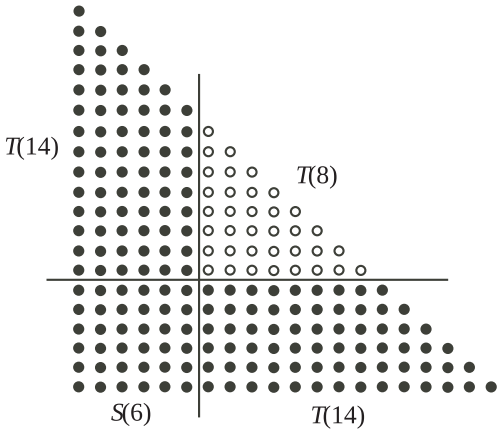
Note that the coins of the two $T(2)$-triangles at the end of the $T(8)$-triangle fill in the empty spaces of a $T(3)$-triangle to complete the $S(6)$-square. In other words, $T(8) = S(6) + 2T(2) - T(3)$ and $T(3) = 2T(2)$ imply that $S(6) = T(8)$.

The following diagram shows that (I) is true in general:



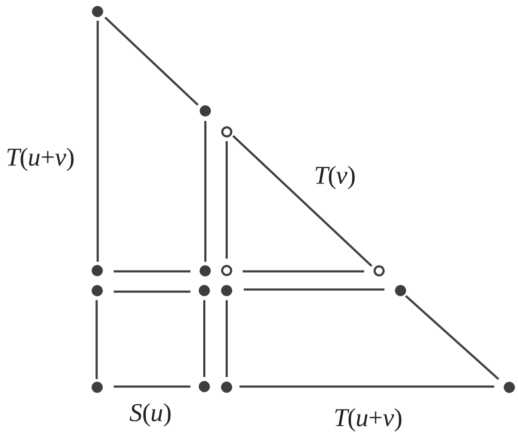
Note that the coins of the two $T(y)$-triangles at the end of the $T(x+2y+1)$-triangle fill in the empty spaces of a $T(x)$-triangle to complete the resulting $S(x + y + 1)$-square.

Next, we show that $S(6) = T(8)$ implies that $T(20) = 2T(14)$:



The coins of the $S(6)$-square at the bottom of the $T(20)$-triangle can be stacked on the coins of the $T(8)$-triangle to form two overlapping copies of $T(14)$-triangle. That is, $T(20) = 2T(14) - T(8) + S(6)$ and $S(6) = T(8)$ imply that $T(20) = 2T(14)$.

The following diagram shows that (II) is true in general:



Note that the coins of the $S(u)$-square at the bottom of the $T(u + 2v)$-triangle can be stacked on the coins of the $T(v)$-triangle to form two overlapping $T(u + v)$-triangles.

These diagrams show that facts (I) and (II) are true. These facts, combined with the algebraic work above, generate the sequence of solutions.

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THE OLYMPIAD CORNER

No. 289

R.E. Woodrow

We begin this number of the *Corner* with problems from the 3rd and 4th grade of the Croatian National Mathematical Competition. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam and to Željko Hanjš, Zagreb, for making them available for our use.

CROATIAN MATHEMATICAL COMPETITION 2007
National Competition

3rd Grade

1. Let n be a positive integer such that $n + 1$ is divisible by 24.

(a) Prove that n has an even number of divisors (including 1 and n itself).

(b) Prove that the sum of all divisors of n is divisible by 24.

(Simplified from Putnam Competition 1969)

2. In the triangle ABC, with $\angle BAC = 120^\circ$, the bisectors of the angles $\angle BAC$, $\angle ABC$, $\angle BCA$ intersect the opposite sides in the points D, E, F, respectively. Prove that the circle with diameter EF passes through D.
(British Mathematical Olympiad 2005)

3. In triangle ABC the vertex A is equidistant from the circumcentre and the orthocentre. Find the angle $\alpha = \angle BAC$.
(USA proposal for IMO 1989)

4. Ten integers 1, 4, 7, $\dots$, 28 (an arithmetic progression with common difference 3) are arranged in a circle. Let N be the maximum of the 10 sums obtained by adding to any integer its two neighbours on the circle. What is the minimum possible value of N?

4th Grade

1. The same as problem 1 of 3rd Grade.

2. A sequence of positive integers $(a_n)_{n\geq 0}$ is defined recursively by $a_0 = 3$ and $a_n = 2 + a_0a_1\cdots a_{n-1}$ for $n \geq 1$.

(a) Prove that any two distinct terms of the sequence are relatively prime.

(b) Determine a_{2007}.

3. In a $5 \times n$ table, where n is a positive integer, each 1×1 cell is painted either red or blue. Find the smallest possible n such that, for any painting of the table, one can always choose three rows and three columns for which the 9 cells in their intersection all have the same colour.

4. In acute triangle ABC let A_1, B_1 and C_1 be the midpoints of the sides BC, CA, and AB, respectively. The circumcircle of ABC has centre O and radius 1. Prove that

$$\frac{1}{|OA_1|} + \frac{1}{|OB_1|} + \frac{1}{|OC_1|} \ \geq\ 6\,.$$

Next we look to the 51st National Mathematics Olympiad in Slovenia and the Selection Examinations for IMO 2007. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam for collecting them for us.

51st NATIONAL MATHEMATICAL OLYMPIAD IN SLOVENIA

Selection Examinations for the IMO 2007

First Selection Examination, December 2006

1. Show that the inequality $(1 + a^2)(1 + b^2) \geq a(1 + b^2) + b(1 + a^2)$ holds for any pair of real numbers a and b.

2. Prove that any triangle can be decomposed into n isosceles triangles for every positive integer $n \geq 4$.

3. Let ABC be a triangle with $AC < BC$ and denote its circumcircle by Γ. Let E be the midpoint of the arc AB that contains the point C and let D be a point on the segment BC, such that $BD = AC$. The line DE meets the circle Γ again in F. Prove that A, B, C, and F are the vertices of an isosceles trapezoid.

Second Selection Examination, February 2007

1. Every point in the plane with positive integer coordinates (x, y) such that $x \leq 19$ and $y \leq 4$ is coloured green, red, or blue. Prove that there exists a rectangle with sides parallel to the coordinate axes and with all four vertices of the same colour.

2. The circles Γ_1 and Γ_2 of different radii meet at A_1 and A_2. Let t be the common tangent of the two circles, such that the distance from t to A_1 is shorter than the distance from t to A_2. Let B_1 and B_2 be the points at which t touches Γ_1 and Γ_2, respectively.

Let Γ_3 and Γ_4 be the circles with radii $|A_1B_1|$ and $|A_1B_2|$ and the centre A_1. The circles Γ_1 and Γ_3 meet again at C_1, while the circles Γ_2 and Γ_4 meet again at C_2. Denote the intersection of the lines B_1C_1 and B_2C_2 by D and let E be the intersection of B_1C_1 and Γ_4 which lies on the same side of the line B_2C_2 as C_1.

Show that A_1D is perpendicular to EC_2.

3. Find a positive integer n such that $n^2 - 1$ has exactly 10 positive divisors. Show that $n^2 - 4$ cannot have exactly 10 positive divisors for any positive integer n.

Third Selection Examination, March 2007

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x^2(z^2+1)+f(y)(z+1)) \;=\; 1-f(z)(x^2+f(y))-z(1+z)x^2+2f(y)$$

for all real numbers x, y, and z.

2. Let

$$x \;=\; 0.a_1a_2a_3a_4\ldots \quad \text{and} \quad y \;=\; 0.b_1b_2b_3b_4\ldots$$

be the decimal representations of two positive real numbers. The equality $b_n = a_{2^n}$ holds for all positive integers n. Given that x is a rational number, show that y is also a rational number.

3. Let $ABCD$ be a trapezoid with AB parallel to CD and $|AB| > |CD|$. Let E and F be the points on the segments AB and CD, respectively, such that $\dfrac{|AE|}{|EB|} = \dfrac{|DF|}{|FC|}$. Let K and L be two points on the segment EF such that

$$\angle AKB \;=\; \angle DCB \quad \text{and} \quad \angle CLD \;=\; \angle CBA\,.$$

Show that K, L, B, and C are concyclic.

Next we look at some problems of the Correspondence Mathematical Competition in Slovakia 2006/2007. These are arranged by students of Comenius University in Bratislava, with support of the Slovak Mathematical Olympiad Committee. In a year there are two rounds of competitions, each round consisting of three series of problems, some for first and second year high school students (1–7 in a set), some for older students (5–11), and some intended as IMO preparation (10–14). We give the fourteen problems of the first set from the first round. The organizers often use problems from other contests and note the source where possible. Thanks again go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

CORRESPONDENCE MATHEMATICAL COMPETITION IN SLOVAKIA 2006/7
First Round

First Set

1. Some pigeons and sparrows are sitting on a fence. Five sparrows fly away and there remain two pigeons for each sparrow. Then 25 pigeons fly away and there remain three sparrows for each pigeon. Find the initial numbers of sparrows and pigeons.

2. There are six dominoes on the table. They are arranged to form a perimeter of a square of size 4×4. Fnd the smallest possible number of spots on all the dominoes together. *Remark*: There are from 0 to 6 spots on each half of the domino. The full set of dominoes contains all 28 possible pieces. [*Ed.: This is how we received this problem and an interpretation of the problem yielding an interesting solution would be much appreciated.*]

3. We have eight cubes with digits 1, 2, 3, 4, 5, 6, 7, 9 (each cube has one digit written on one of its faces). In how many ways can we create four two-digit primes from the cubes?

4. A nine-member committee was formed to select a chief of the KMS. There are three candidates for the chief. Each member of the committee orders the candidates and gives 3 points to the first one, 2 points to the second one and 1 point to the last one. After summing the points of the candidates no two candidates had the same number of points, hence the order of the candidates was clear. Someone noticed that if every member of the committee selected only one candidate, then the resulting order of the candidates was reversed. How many points did the candidates get?

5. (a) Find all positive integers n such that both of the numbers $2^n - 1$ and $2^n + 1$ are primes.

(b) Find all primes p such that both of the numbers $4p^2 + 1$ and $6p^2 + 1$ are primes.

6. Find all positive integers n such that $n + 200$ and $n - 269$ are cubes of integers.

7. There were 33 children at a camp. Every child answered two questions: "How many other children at camp have the same first name as you?" and "How many other children at camp have the same family name as you?" Among the answers each of the numbers from 0 to 10 occurred at least once. Show that there were at least two children at camp with the same first name and the same family name.

(Mathematical Contests 1997–1998, 1.18 Russia, 29/95)

8. There are $2n$ white and $2n$ black balls in a row. Prove that, whatever order they are in, we can always find $2n$ consecutive balls of which exactly n are white.

9. Find all triples of integers x, y, z satisfying

$$2^x + 3^y = z^2 .$$

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10. Numismatist Christian has 241 coins with total value of 360 talers. (The value of a coin in talers is a positive integer.) Can Christian be sure that he can divide his coins into three piles all of equal value?

(Ukraine 2005)

11. There are n people living on an island. One day their leader decided that all islanders (including himself) will make and wear a necklace composed of one-coloured stones (at least zero stones per necklace). Two islanders are to have at least one stone of the same colour in their necklaces if and only if they are friends.

(a) Prove that the islanders can fulfill their leader's orders.

(b) At least how many colours of stones are needed to fulfill the order, regardless of what the friendships on the island are?
(Belarus 2001)

12. We are given an acute triangle ABC with circumcentre O. Let T be the circumcentre of AOC. Let M be the midpoint of AC. The points D and E lie on the lines AB and CB respectively in such a way that the angles MDB and MEB are equal to the angle ABC. Prove that the lines BT and DE are perpendicular.

13. A line passing through the centroid T of the triangle ABC meets the side AB at P and the side CA at Q. Prove that

$$4 \cdot PB \cdot QC \;\leq\; PA \cdot QA .$$

(R.B. Manfrino: Inequalities, 111/3.29, Spain 1998)

14. Prove that if integers x, y each greater than 1 satisfy $2x^2 - 1 = y^{15}$, then 5 divides x. Can you find such integers x and y?

(Russia 2004/05)

Next we turn to the 57th Latvian Mathematical Olympiad 2007 and the problems for Grade 11 and Grade 12 for the 3rd round. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for obtaining them for the *Corner*.

57th LATVIAN MATHEMATICAL OLYMPIAD 2007

Problems of the 3rd Round

Grade 11

1. For a positive integer n

(a) can the sums of digits of n and $n + 2007$ be equal?

(b) can the sums of digits of n and $n + 199$ be equal?

2. Do there exist three quadratic trinomials such that each of them has at least one root, but the sum of any two quadratic trinomials doesn't have any roots?

3. Each side of a sheet of paper is partitioned into 3 polygons. On one side one of the polygons is coloured white, another red, and the third one green. Prove that on the other side of the sheet it is possible to colour one of the polygons white, another red, and the third one green in such a way that at least one third of the area of the paper sheet is coloured with the same colour on both sides.

4. In $\triangle ABC$ the point K lies on median AM and $\angle BAC + \angle BKC = 180°$. Prove that $AB \cdot KC = AC \cdot KB$.

5. For a sequence of real numbers a_1, a_2, a_3, $\ldots$ we have $a_{11} = 4$, $a_{22} = 2$, and $a_{33} = 1$. In addition the relation

$$\frac{a_{n+3} - a_{n+2}}{a_n - a_{n+1}} = \frac{a_{n+3} + a_{n+2}}{a_n + a_{n+1}}$$

holds for each n. Prove that

(a) $a_i \neq 0$ for each i,

(b) the sequence is periodic,

(c) $a_1^k + \cdots + a_{100}^k$ is a square of an integer for each positive integer k.

Grade 12

1. What can be the values of nonnegative real numbers a and b, if it is known that equations $x^2 + a^2x + b^3 = 0$ and $x^2 + b^2x + a^3 = 0$ have a common real root?

2. At each vertex of an n-gonal prism the number $+1$ or -1 is written, and the product of the numbers on each face of the prism is -1. Can $n = 4$? Can $n = 10$?

3. Solve the system of equations

$$\begin{cases} \sin^2 x + \cos^2 y &= y^2\,, \\ \sin^2 y + \cos^2 x &= x^2\,. \end{cases}$$

4. Two circles w_1 and w_2 intersect in points A and B. Line t_1 is drawn through point B with the other intersection point with w_1 being C and the other intersection point with w_2 being E. Line t_2 is drawn through point B the with other intersection point with w_1 being D and the other intersection point with w_2 being F. Point B lies between C and E and between D and F. Midpoints of segments CE and DF are denoted by M and N. Prove that triangles ACD, AEF, and AMN are similar.

5. The set of all positive integers is partitioned into several parts so that each positive integer belongs to exactly to one part and each part contains infinitely many integers. Can this be done so that one part contains a multiple of each positive integer? Give the answer if

(a) there are a finite number of parts,

(b) there are an infinite number of parts.

Next we turn to the Final Round of the Finnish National High School Mathematics Competition 2007. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam for collecting them for our use.

FINNISH NATIONAL HIGH SCHOOL MATHEMATICS COMPETITION 2007

Final Round

Helsinki, February 2, 2007

1. Show that when a prime number is divided by 30, the remainder is either a prime number or 1. Is a similar claim true when the divisor is 60 or 90?

2. Determine the number of real roots of the equation

$$x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \frac{5}{2} \;=\; 0\,.$$

3. There are five points in the plane, no three of which are collinear. Show that some four of these points are the vertices of a convex quadrilateral.

4. The six offices of the City of Salavaara are to be conected to each other by a communication network which utilizes modern picotechnology. Each of the offices is to be connected to all the other ones by direct cable connections.

Three operators compete to build the connections, and there is a separate competition for every connection. When the network is finished one notices that the worst has happened: the systems of the three operators are incompatible. So the city must reject connections built by two of the operators, and these are to be chosen so that the damage is minimized. What is the minimal number of offices which still can be connected to each other, possibly through intermediate offices, in the worst possible case?

5. Show that there exists a polynomial $P(x)$ with integer coefficients such that the equation $P(x) = 0$ has no integer solutions but for each positive integer n there is an integer m such that $n \mid P(m)$.

To complete the problem sets for this number we give the IX Olimpiada Mathemático de Centroamérica y El Caribe 2007. Thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam for collecting them for us.

IX OLIMPIADA MATHEMÁTICO DE CENTROAMÉRICA Y EL CARIBE 2007

First Day (Tuesday, June 5, 2007)

1. The OMCC is an annual mathematical competition. The ninth olympiad takes place in the year 2007. Which positive integers n divide the year in which the n^{th} olympiad takes place?

2. Let ABC be a triangle; D, E points on the sides AC, AB, respectively, such that the lines BD, CE, and the angle bisector of angle A concur at an interior point P of the triangle. Prove that there is a circle tangent to the four sides of the quadrilateral $ADPE$ if and only if $AB = AC$.

3. Let S be a finite set of integers. For any two integers p, q in S with $p \neq q$, there are integers a, b, c in S, not necessarily distinct and with $a \neq 0$, such that the polynomial $F(x) = ax^2 + bx + c$ satisfies $F(p) = F(q) = 0$. Determine the maximum number of elements the set S can have.

Second Day (Wednesday, June 6, 2007)

4. The inhabitants of a certain island speak a language in which every word can be written with the following letters: a, b, c, d, e, f, g. A word is said to *produce* another one if the second word can be formed from the first one by applying any of the following rules as many times as needed:

(i) Replace a letter by two letters according to one of the substitutions

$$a \to bc\,,\ b \to cd\,,\ c \to de\,,\ d \to ef\,,\ e \to fg\,,\ f \to ga\,,\ g \to ab\,.$$

(ii) If only one letter is between two letters that are the same, these two letters can be eliminated. For example, $dfd \to f$.

As another example, $cefed$ produces $bfed$, since $cafed \to cbcfed \to bfed$.

Prove that every word on this island produces any other word.

5. Given two nonnegative integers m and n with $m > n$, we say that m *ends in* n if one can erase some consecutive digits from the left of m to obtain n. For example, 329 ends in 9 and in 29. Determine how many three-digit numbers end in the product of their digits.

6. Let A and B be points on the circle Γ such that the lines PA and PB are tangent to Γ for an exterior point P. Let M be the midpoint of AB. The perpendicular bisector of AM intersects Γ at C which is interior to $\triangle ABP$, the line AC intersects the line PM at G, and the line PM intersects Γ at D, which is exterior to the triangle $\triangle ABP$. If BD is parallel to AC, prove that G is the point in which the medians of $\triangle ABP$ concur.

We pick up again with solutions to problems of the Croatian Mathematical Olympiad 2006, National Competition, 4$^{\text{th}}$ Grade [2009 : 293–294].

2. Let k and n be positive integers. Prove that

$$(n^4 - 1)(n^3 - n^2 + n - 1)^k \; + \; (n+1)n^{4k-1}$$

is divisible by $n^5 + 1$.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's solution.

We fix n and use induction on k.

Let $P(k) = (n^4 - 1)(n^3 - n^2 + n - 1)^k + (n+1)n^{4k-1}$. We have that $(n^5 + 1) \mid P(1)$, since

$$\begin{aligned} P(1) &= (n^4 - 1)(n^3 - n^2 + n - 1) + (n+1)n^3 \\ &= (n^5 + 1)(n^2 - n + 1)\,, \end{aligned}$$

Now assume that $(n^5 + 1) \mid P(k)$ for some integer $k > 0$. Then

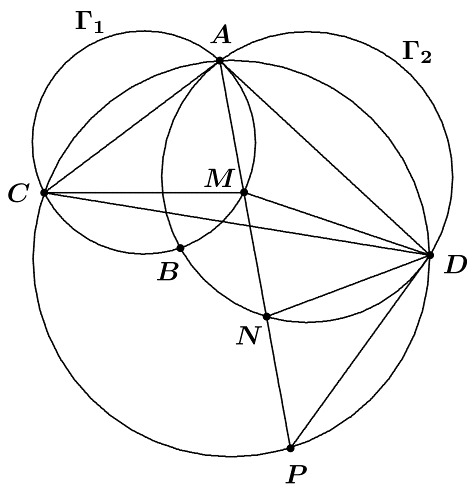
$$\begin{aligned} &P(k+1) \\ &\quad = (n^4 - 1)(n^3 - n^2 + n - 1)^k(n^3 - n^2 + n - 1) + (n+1)n^{4k+3} \\ &\quad = [P(k) - (n+1)n^{4k-1}](n^3 - n^2 + n - 1) + (n+1)n^{4k+3} \\ &\quad = P(k)(n^3 - n^2 + n - 1) + (n+1)n^{4k-1}(n^4 - n^3 + n^2 - n + 1) \\ &\quad = P(k)(n^3 - n^2 + n - 1) + (n^5 + 1)n^{4k-1}\,, \end{aligned}$$

and consequently $(n^5 + 1) \mid P(k+1)$.

We conclude that $(n^5 + 1) \mid P(k)$ for each integer $k \geq 1$.

3. The circles Γ_1 and Γ_2 intersect at the points A and B. The tangent line to Γ_2 through the point A meets Γ_1 again at C and the tangent line to Γ_1 through A meets Γ_2 again at D. A half-line through A, interior to the angle $\angle CAD$, meets Γ_1 at M, meets Γ_2 at N, and meets the circumcircle of $\triangle ACD$ at P. Prove that $|AM| = |NP|$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We give the solution of Amengual Covas.



We use the theorem that the angle between a tangent and a chord of a circle is equal to the angle in the segment on the opposite side of the chord. Thus, $\angle ACM = \angle DAM = \angle DAN$ and $\angle CAN = \angle ADN$, making $\triangle ACM$ similar to $\triangle DAN$, so that $\dfrac{AM}{AC} = \dfrac{DN}{AD}$.

Since quadrilateral $ACPD$ is cyclic, on chord DA we have $\angle APD = \angle ACD$ and on chord CP we have

$$\begin{aligned}\angle PDC &= \angle CAP \\ &= \angle CAN = \angle ADN\,.\end{aligned}$$

Hence, $\angle PDC - \angle NDC = \angle ADN - \angle NDC$, that is, $\angle PDN = \angle CDA$. It follows that $\triangle NPD$ is similar to $\triangle ACD$, hence $\dfrac{NP}{AC} = \dfrac{DN}{AD}$.

From $\dfrac{AM}{AC} = \dfrac{DN}{AD}$ and $\dfrac{NP}{AC} = \dfrac{DN}{AD}$ we get $AM = NP$, as desired.

Next we turn to the Balkan Mathematical Olympiad 2006, Nicosia, Cyprus, Greece, given at [2009 : 294].

1. (*Greece*) Let a, b, and c be real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}\,.$$

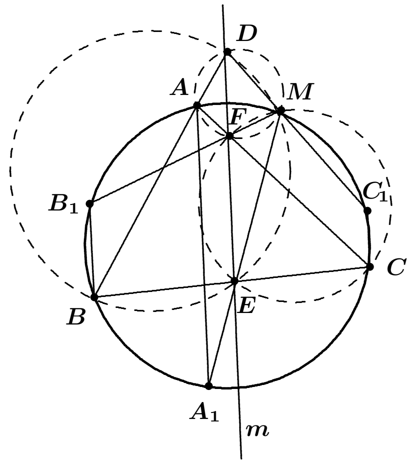
Solved by Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Comăneşti, Romania. Comments by Mohammed Aassila, Strasbourg, France; and Michel Bataille, Rouen, France. We give Bataille's comment.

The inequality does not hold if we take $a = b = -2$ and $c = 1$. Thus we assume that the intended hypothesis is a, b, $c > 0$. If this is actually the case, then the problem is not new: it is problem 2362 [1998 : 304; 1999 : 375].

2. (*Greece*) Let ABC be a triangle and m a line which intersects the sides AB and AC at interior points D and F, respectively, and intersects the line BC at a point E such that C lies between B and E. The lines through points A, B, C and parallel to the line m intersect the circumcircle of triangle ABC again at the points A_1, B_1, C_1, respectively. Prove that the lines A_1E, B_1F, and C_1D are concurrent.

Solution by Michel Bataille, Rouen, France.

More generally, the result holds whenever m is any transversal of $\triangle ABC$ (see the figure).



Since D, E, F are points on the lines AB, BC, CA, distinct from the vertices, Miquel's theorem tells us that the circumcircles of $\triangle ADF$, $\triangle BDE$, and $\triangle CEF$ have a common point, M. Applying the same theorem to the points B, C, E on the lines AD, AF, DF, respectively, we see that M must be a point on the circumcircle of $\triangle ABC$.

Now, denoting by $\angle(\ell, \ell')$ the directed angle between the lines ℓ and ℓ', we have

$$\begin{aligned}\angle(MA_1, ME) &= \angle(MA_1, m) + \angle(m, ME) \\ &= \angle(A_1M, A_1A) + \angle(EF, EM) \\ &= \angle(CM, CA) + \angle(CF, CM)\,,\end{aligned}$$

where we have used the fact that A, C, M, A_1 are concyclic and also the fact that E, C, F, M are concyclic.

Thus, $\angle(MA_1, ME) = 0$, which means that A_1E passes through M. Similarly, B_1F and C_1D pass through M, and the result follows.

3. (*Romania*) Find all triples of positive rational numbers (m, n, p) such that each of the numbers

$$m + \frac{1}{np}\,, \qquad n + \frac{1}{pm}\,, \qquad p + \frac{1}{mn}$$

is an integer.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We use Bataille's write-up.

We show that the solutions for (m, n, p) are the triples $(\frac{1}{2}, \frac{1}{2}, 4)$, $(\frac{1}{2}, 1, 2)$ and their permutations, and the triple $(1, 1, 1)$.

It is easily checked that these triples are solutions.

Conversely, for a solution (m, n, p) define positive integers a, b, c by

$$a = m + \frac{1}{np}, \qquad b = n + \frac{1}{pm}, \qquad c = p + \frac{1}{mn}.$$

Since $mnp + 1 = anp = bpm = cmn$, we have $abc(mnp)^2 = (mnp + 1)^3$. Upon setting $mnp = \frac{u}{v}$ where u, v are coprime positive integers, we have $abcu^2v = (u + v)^3$.

Now, any prime number dividing u^2v must divide u or v, hence cannot divide $u + v$ (since otherwise it would divide both u and v). Thus, u^2v and $(u+v)^3$ are coprime and since $abc = \frac{(u+v)^3}{u^2v}$, it follows that $(u+v)^3 = abc$ and $u^2v = 1$. Thus, $u = v = 1$, $abc = 8$, $mnp = 1$.

Assuming $a \le b \le c$, the only possibilities for (a, b, c) are $(1, 1, 8)$, $(1, 2, 4)$, $(2, 2, 2)$ which lead to $(\frac{1}{2}, \frac{1}{2}, 4)$, $(\frac{1}{2}, 1, 2)$, $(1, 1, 1)$ for (m, n, p). The result follows.

Now we turn to solutions from our readers to problems of the Finnish Mathematical Olympiad 2006, Final Round, given at [2009 : 295].

1. Determine all pairs (x, y) of positive integers such that

$$x + y + xy \;=\; 2006\,.$$

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give Kandall's write-up.

Adding 1 to both sides of the given relation, we obtain

$$(1 + x)(1 + y) \;=\; 2007\,.$$

The prime factorization of 2007 is $3^2 \cdot 223$, so 2007 has exactly six positive divisors: 1, 3, 9, 223, 669, 2007. Consequently, the only admissible products $(1+x)(1+y)$ are $3\cdot 669$, $9\cdot 223$, $223\cdot 9$ and $669\cdot 3$. Thus, the only solutions (x, y) are $(2, 668)$, $(8, 222)$, $(222, 8)$, and $(668, 2)$.

2. For all real numbers a, prove that

$$3\left(1 + a^2 + a^4\right) \;\ge\; \left(1 + a + a^2\right)^2$$

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Curtis.

Since $a^4 + a^2 + 1 = (a^2 + a + 1)(a^2 - a + 1)$, the claimed inequality is equivalent to

$$2(a-1)^2(a^2+a+1) \ \geq \ 0\,,$$

which is true, as $a^2 + a + 1 = (a + \frac{1}{2})^2 + \frac{3}{4}$ for all real numbers a.

3. The numbers p, $4p^2 + 1$, and $6p^2 + 1$ are primes. Determine p.

Solved by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Cománeşti, Romania. We give Alt's write-up.

First consider primes $p = 2$, 3, and 5.
If $p = 2$ then $4p^2 + 1 = 17$ is prime, but $6p^2 + 1 = 25$ is not prime.
If $p = 3$ then $4p^2 + 1 = 37$ is prime, but $6p^2 + 1 = 55$ is not prime.
If $p = 5$ then $4p^2 + 1 = 101$ and $6p^2 + 1 = 151$ are both primes.
Now let p be a prime greater than 5. Since

$$\begin{aligned} 4p^2+1 &= 5p^2-(p^2-1) \ \equiv \ -(p^2-1) \pmod 5\ , \\ 6p^2+1 &= 5(p^2-p-1)+(p+2)(p+3) \\ &\equiv (p+2)(p+3) \pmod 5 \end{aligned}$$

and

$$-(p-1)p(p+1)(p+2)(p+3) \ \equiv \ 0 \pmod 5\ ,$$

it follows that

$$p(4p^2+1)(6p^2+1) \ \equiv \ 0 \pmod 5\ .$$

Then $(4p^2+1)(6p^2+1) \equiv 0 \pmod 5$, because p and 5 are coprime. Hence, $4p^2+1$ or $6p^2+1$ is a composite number, because each is greater than 5 and one of them is divisible by 5.

Thus, the only solution to the problem is $p = 5$.

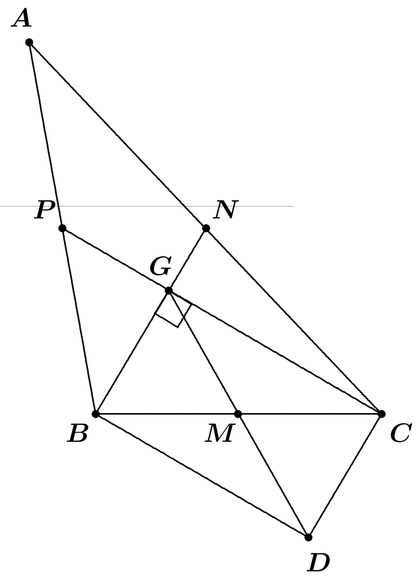
4. Prove that if two medians of a triangle are perpendicular, then the triangle whose sides are congruent to the medians of the original triangle is a right triangle.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Cománeşti, Romania. We give the two solutions by Amengual Covas.

First Solution: Let ABC be the given triangle, and let M, N, and P be the midpoints of the sides BC, CA, and AB, respectively. Let G be the centroid of $\triangle ABC$ and let D be symmetric to G with respect to M.

Since segments BC and GD bisect each other, the quadrilateral $BDCG$ is a parallelogram, so we have

$$GD = 2 \cdot GM = \frac{2}{3}AM\,,$$
$$DC = BG = \frac{2}{3}BN\,,$$
$$CG = \frac{2}{3}CP\,.$$



Hence, $\triangle CGD$ is similar to the triangle whose sides are congruent to the medians of $\triangle ABC$.

Thus, it suffices to prove that $\triangle CGD$ is a right triangle if two medians of $\triangle ABC$ are perpendicular.

Without loss of generality take $BN \perp CP$. Then $\angle BGC = 90^\circ$, so $BDCG$ is a rectangle.

Then $\angle GCD = 90^\circ$ and $\triangle GCD$ is a right triangle, as required.

Second Solution: Let m_a, m_b, m_c be the medians of the triangle to the sides a, b, c, respectively. Assume without loss of generality that m_b and m_c are perpendicular and we use the fact that this is equivalent to the relation $b^2 + c^2 = 5a^2$ to obtain

$$\begin{aligned} m_b^2 + m_c^2 &= \frac{1}{4}(2c^2 + 2a^2 - b^2) + \frac{1}{4}(2a^2 + 2b^2 - c^2) \\ &= a^2 + \frac{1}{4}(b^2 + c^2) = \frac{9}{4}a^2 = \frac{1}{4}(2(b^2 + c^2) - a^2) = m_a^2\,. \end{aligned}$$

The conclusion now follows from the converse of the Pythagorean Theorem.

To complete this *Corner* we look at a solution to a problem of the Estonian Mathematical Olympiad 2005–2006, Final Round [2009 : 375–376].

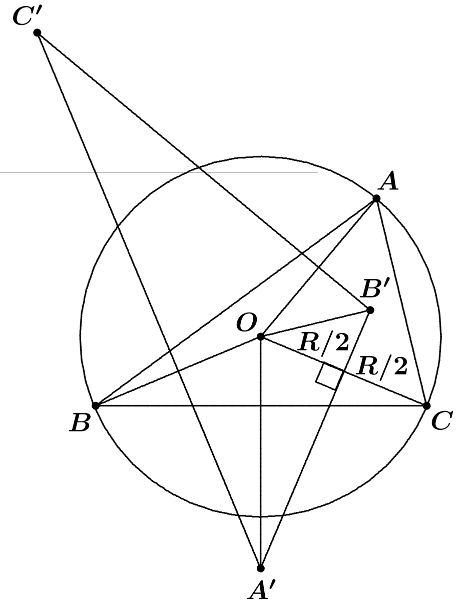
4. The acute triangle ABC has circumcentre O and triangles BCO, CAO, and ABO have circumcentres A', B', and C', respectively. Prove that the area of triangle ABC does not exceed the area of triangle $A'B'C'$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give the write-up of Amengual Covas.

Let R be the circumradius of $\triangle ABC$. Since $\triangle ABC$ is acute, O is inside $\triangle ABC$, so $\angle A'OC = \angle BAC$, $\angle COB' = \angle ABC$, and $\angle AOC' = \angle BCA$. We also have that $A'B'$ is the perpendicular bisector of segment OC.

Hence, $OA' = \dfrac{R/2}{\cos\angle A'OC} = \dfrac{R}{2\cos A}$ and $OB' = \dfrac{R/2}{\cos\angle COB'} = \dfrac{R}{2\cos B}$.

Therefore, with brackets denoting the area of the enclosed figure,



$$\begin{aligned}[OA'B'] &= \frac{1}{2}OA'\cdot OB'\cdot\sin\angle A'OB' \\ &= \frac{1}{2}\cdot\frac{R}{2\cos A}\cdot\frac{R}{2\cos B}\cdot\sin(A+B) \\ &= \frac{1}{8}R^2\frac{\sin C}{\cos A\cos B},\end{aligned}$$

since $A + B = 180^\circ - C$.

Similarly, we obtain

$$\begin{aligned}[OB'C'] &= \frac{1}{8}R^2\frac{\sin A}{\cos B\cos C}, \\ [OC'A'] &= \frac{1}{8}R^2\frac{\sin B}{\cos C\cos A}.\end{aligned}$$

The following are known:

$$\begin{aligned}\sin 2A + \sin 2B + \sin 2C &= 4\sin A\sin B\sin C, &(1)\\ [ABC] &= 2R^2\sin A\sin B\sin C, &(2)\\ \cos A\cos B\cos C &\le \frac{1}{8}, &(3)\end{aligned}$$

(see, respectively, Formula 120, p. 105 and Formula 254, p. 178 of *Relations entre les éléments d'un triangle*, Librairie Nony, Paris, 1893, and item 2.23, p. 25, of O. Bottema et al., *Geometric Inequalities*, Groningen, 1969).

We thus obtain

$$\begin{aligned}[A'B'C'] &= [OA'B'] + [OB'C'] + [OC'A'] \\ &= \frac{1}{8}R^2\cdot\frac{\sin A\cos A + \sin B\cos B + \sin C\cos C}{\cos A\cos B\cos C} \\ &= \frac{1}{16}R^2\cdot\frac{\sin 2A + \sin 2B + \sin 2C}{\cos A\cos B\cos C} \\ &= \frac{1}{4}R^2\cdot\frac{\sin A\sin B\sin C}{\cos A\cos B\cos C} = \frac{1}{8}\frac{[ABC]}{\cos A\cos B\cos C} \ge [ABC],\end{aligned}$$

as desired. Equality occurs if and only if $\triangle ABC$ is equilateral.

That completes another (somewhat short!) *Corner*, and we're aiming to clear a backlog of material in the next issue. As always we welcome your nice solutions and generalizations.

BOOK REVIEWS

Amar Sodhi

Who Gave You the Epsilon? & Other Tales of Mathematical History
Edited by Marlow Anderson, Victor Katz, and Robin Wilson
The Mathematical Association of America, Washington, DC, 2009
ISBN-13: 978-0-88385-569-0, hardcover, 429+x pages, US$65.50
Reviewed by ***Jeff Hooper****, Acadia University, Wolfville, NS*

For more than one hundred years, the Mathematical Association of America (MAA) has been publishing journals geared to a general mathematics readership. This present volume, a sequel to the editors' earlier excellent *Sherlock Holmes in Babylon*, consists of a collection of articles culled from MAA journals and dealing with aspects of the History of Mathematics, in particular considering developments in the nineteenth and twentieth centuries.

The book's articles are arranged by topic into chapters on Analysis; Geometry, Topology, and Foundations; Algebra and Number Theory; and a short final one consisting of Surveys. Throughout each section, the editors' choices of articles focus on those which examine the development of critical concepts. For a number of topics there are multiple articles which examine the same concept from different perspectives. In some cases the articles selected on a topic were written years apart, and reading them provides an interesting glimpse into the way the ideas and approaches have evolved.

The Analysis chapter begins with an article by Judith V. Grabiner which gives the book its title. Many readers will have battled with the $\varepsilon - \delta$ definitions of limit and continuity which underpin calculus, and Grabiner takes us on a fascinating exploration of Cauchy's rigorization of the ideas of Newton and Leibniz. In particular she focuses on why the desire for rigor arose in the first place. Cauchy's predecessors were content to work with the less-rigorous version of calculus.

Further articles explore the evolution of the function concept, of Green's work on electricity and magnetism, and Stokes' Theorem. Also included are excellent biographical sketches on the significant mathematical contributions of two pioneer mathematicians, looking at the lives of Sofya Kovalevsky and David Blackwell and the myriad societal challenges overcome by each.

The nineteenth century was a pivotal one in the development of modern geometry, so it should be no surprise to find included in the Geometry, Topology, and Foundations chapter articles on Euclid's parallel postulate and the development of non-Euclidean Geometry. Among other topics here, the articles also explore the developing notion of connectedness, as well as the far-reaching concepts of homotopy and homology. The nature of infinite sets is also examined, as are the remarkable properties of Cantor sets and functions.

The last two centuries witnessed enormous progress in algebra and number theory as well, and a number of key ideas are highlighted here, including Hamilton's quaternions, Galois' discoveries, the development of the group concept and the hunt for finite simple groups, and the history of the prime number theorem. The lives of Galois, Ramanujan, and Emmy Noether are also examined.

The final closing chapter examines the development of mathematics from a wider point of view, including three survey articles on mathematics (written in 1900, 1951, and 2000), as well as a short account of the 1900 International Congress of Mathematicians in Paris, now famous for Hilbert's address on the major open problems in mathematics, by an American delegate, G.B. Halsted. Each of these surveys gives a panoramic view of areas under active investigation at that time. Since they are describing research frontiers, these surveys necessarily consider more sophisticated mathematics. Nevertheless, they still provide useful general perspectives of how math is seen to be developing at each point in time.

So, if you are intrigued by the historical development of mathematics during the past 200 years, then there is a wealth of material collected for you in these 41 articles. I highly recommend it.

The Princeton Companion to Mathematics
Edited by Timothy Gowers with associate editors
June Barrow-Green and Imre Leader
Published by Princeton University Press, 2008
ISBN 978-0-691-11880-2, hardcover, 1034+xx pages, US$99.00
Reviewed by ***R.W. Richards****, Sir Wilfred Grenfell College, Corner Brook, NL*

This large (more than thousand page) opus aims to provide an overview of mathematics as it is practiced today. It is not intended as a dictionary of mathematics, nor an encyclopedia. The content covers a large spectrum of current areas of interest in pure mathematics, and is aimed at a readership of non-experts. There are more than 130 contributors from many American, European, and other research institutions. The reader with some experience in mathematics will recognize many of the names.

It is an ambitious project to provide a self-contained exposition accessible to those not familiar with a field. In this objective, this volume succeeds remarkably well, although a certain level of mathematical sophistication and some patience is required. As the intention is to provide a look at areas of current interest in mathematics, necessarily some topics are omitted, or are only covered briefly. For example, many elementary topics are not discussed, and calculus is only approached in a historical context. At the other end of the spectrum, many areas of research interest are not mentioned, or are only briefly described. For example, a reader interested in Moufang Loops will have to look somewhere else, and a reader interested in Category

Theory will only find a very basic description of that discipline. But again, this is not intended as a dictionary or an encyclopedia and the topics it does cover it does, for the most part, in an interesting and readable way. The editors have tried to present the material so that it is accessible to as large a (mathematically educated) readership as possible. As they state in the preface, they, as a policy, did not include material that they themselves did not understand.

The book is divided into eight parts. The core is in Part IV, which is a collection of twenty six essays on branches of current pure mathematics, covering major areas such as Number Theory, Topology, Geometry, Algebra, Analysis, etc. These essays do achieve the aim of opening large areas to the interested reader. For each there is suggested further reading once the reader's appetite is whetted.

It is preceded, in Part III, by a section of shorter descriptions of mathematical concepts, many of which are then used in the section that follows. These concepts may or may not be familiar to the reader, so a certain amount of jumping back and forth may be necessary.

Part V is devoted to brief entries of important problems of mathematics, some of which were significant in the development of branches of mathematics (Fermat's Last Theorem and the Insolubility of the Quintic), and some of which are major breakthroughs of recent research (the Poincaré Conjecture).

The book opens with an attempt at a current definition of mathematics and a description of what mathematicians do. This is followed by several essays on the development of the fundamental ideas of mathematics – for example the growth of abstraction and the need for rigor.

Near the end of the book, there is a series of short biographies of mathematicians of historical interest, arranged chronologically from Pythagoras to Abraham Robinson and Nicolas Bourbaki. The concluding sections concern mathematics in a broader intellectual, social, and cultural context

At a list price of CD$118.95 (and I notice online discounts) this book is a wonderful bargain. Anyone with an interest in mathematics will welcome this on their bookshelf. There is such a range of topics and lengths of articles, that it can provide many hours of fascinating reading. Were it not for the size and weight, I would even suggest that it contains some good bedtime reading. Further, it would be a valuable holding for any academic library collection.

When do the Curves $xy \equiv 1 \pmod{n}$ and $x^2 + y^2 \equiv 1 \pmod{n}$ Intersect?

Sara Hanrahan and Mizan R. Khan

The second author, MK, would like to dedicate this note to his brother Riaz for introducing him (MK) to the joys of doing elementary mathematics!

Introduction

The figure below illustrates the simple fact that the hyperbola $xy = 1$ does not intersect the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$.

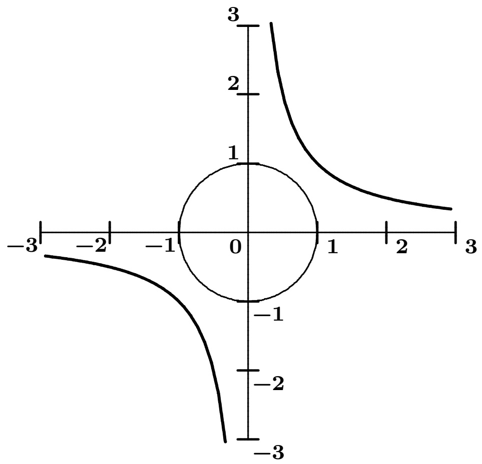


Figure 1. The curves $xy = 1$ and $x^2 + y^2 = 1$

In the course of some investigations in modular arithmetic, we asked ourselves the question of whether the unit modular circle

$$\mathcal{C}_n = \{(x, y) : x, y \in \mathbb{Z}_n \text{ and } x^2 + y^2 \equiv 1 \pmod{n}\}$$

could intersect the modular hyperbola

$$\mathcal{H}_n = \{(x, y) : x, y \in \mathbb{Z}_n \text{ and } xy \equiv 1 \pmod{n}\}.$$

For example, $\mathcal{C}_{17} \cap \mathcal{H}_{17} = \emptyset$, but $\mathcal{C}_{37} \cap \mathcal{H}_{37} \neq \emptyset$, a fact illustrated in the figures on the next page. Note that in our graphs of $\mathcal{C}_n$ and $\mathcal{H}_n$ we introduce the restriction that $0 \le x,\ y \le n - 1$.

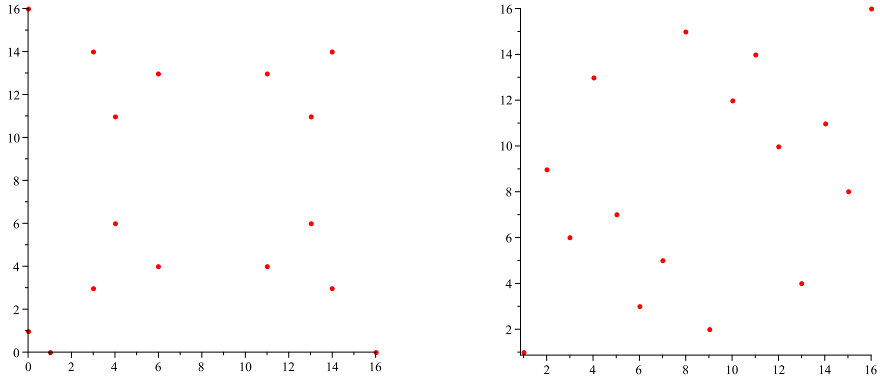


Figure 2. The curves $\mathcal{C}_{17}$ and $\mathcal{H}_{17}$

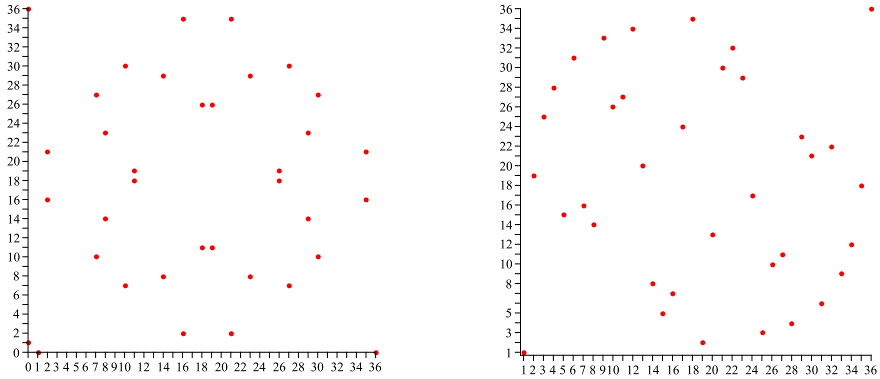


Figure 3. The curves $\mathcal{C}_{37}$ and $\mathcal{H}_{37}$

A more striking visual example of when the curves don't intersect is $n = 787$:

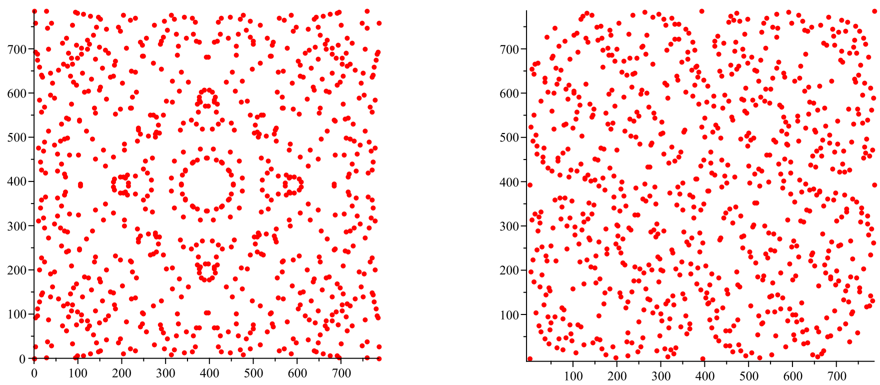


Figure 4. The curves $\mathcal{C}_{787}$ and $\mathcal{H}_{787}$

Eventually we discovered a neat little result about the prime factorization of integers n for which $\mathcal{C}_n \cap \mathcal{H}_n \neq \emptyset$. Specifically, we have the following result.

Theorem 1 It is the case that $\mathcal{C}_n \cap \mathcal{H}_n \neq \emptyset$ if and only if every prime in the canonical factorization of n is congruent to 1 modulo 12.

Background Material

We will prove Theorem 1 by using three basic results in number theory: the Chinese Remainder Theorem, the Law of Quadratic Reciprocity, and a special case of Hensel's lemma. We now give short descriptions of each of these results and refer the reader to [1] for the detailed proofs.

The Chinese Remainder Theorem

The Chinese Remainder Theorem addresses the following type of problem. Is there a positive integer x which when divided by 107 leaves a remainder of 60 and when divided by 256 leaves a remainder of 38? Alternatively, in the language of congruences, can we simultaneously solve

$$x \equiv 60 \pmod{107} \qquad \text{and} \qquad x \equiv 38 \pmod{256}\ ?$$

The Chinese Remainder Theorem says that if the moduli 107 and 256 are relatively prime (which they are), then the answer is yes. The surprise is that the numbers 60 and 38 play no role — only the relationship between 107 and 256 matters.

Theorem 2 (Chinese Remainder Theorem) Let m_1, m_2, ..., m_k be integers such that $\gcd(m_i, m_j) = 1$ whenever $i \neq j$, and let a_1, a_2, ..., a_k be arbitrary integers. Then the congruences

$$x \equiv a_i \pmod{m_i}\ , \quad i = 1, 2, \ldots, k$$

have a common solution, and any two solutions are congruent modulo the product $m = m_1 m_2 \cdots m_k$.

A brief sketch of the proof runs as follows. Let $n_i = m/m_i$. Clearly, $\gcd(n_1, n_2, \ldots, n_k) = 1$. Therefore, by the Extended Euclidean Algorithm, there are integers s_i such that

$$s_1 n_1 + s_2 n_2 + \cdots + s_k n_k \ = \ 1\,.$$

We now take

$$x \ = \ a_1 s_1 n_1 + a_2 s_2 n_2 + \cdots + a_k s_k n_k\,.$$

(See [1], Theorem 3.10, page 53 for more details.)

The Law of Quadratic Reciprocity

The Law of Quadratic Reciprocity deals with questions of when elements of $\mathbb{Z}_n$ have square roots. For example, does the congruence $x^2 \equiv 12 \pmod{97}$ have a solution? The Law of Quadratic Reciprocity provides a very simple relationship when we are working with primes. We first introduce some terminology and the Legendre symbol $(\cdot/p)$. Let p be a prime and a an integer such that $\gcd(a, p) = 1$. We say that a is a *quadratic residue* modulo p if the congruence

$$x^2 \equiv a \pmod{p}$$

has a solution. If this congruence has no solution, then we say that a is a *quadratic nonresidue* modulo p. The Legendre (a/p) is a convenient way to denote whether or not a is a residue or nonresidue modulo p.

Definition For p a prime and a an integer we define the Legendre symbol (a/p) as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } \gcd(a,p) = 1 \text{ and } a \text{ is a quadratic residue mod } p\,, \\ -1, & \text{if } \gcd(a,p) = 1 \text{ and } a \text{ is a quadratic nonresidue mod } p\,, \\ 0, & \text{if } p|a\,. \end{cases}$$

From here on p will always denote an *odd* prime. From the fact that the multiplicative group $\mathbb{Z}_p^*$ is cyclic we can prove that

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}\,,$$

and consequently -1 has a square root in $\mathbb{Z}_p$ if and only if $p \equiv 1 \pmod{4}$. However, for other numbers this is a more difficult question to answer. For example, when is 3 a square root modulo p? An eminently satisfactory answer to such questions is given by the Law of Quadratic Reciprocity.

Theorem 3 (Law of Quadratic Reciprocity) Let p and q be two distinct odd primes. Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}\,.$$

Euler and Lagrange were the first to formulate this law, but Gauss gave the first proof in his masterpiece *Disquisitiones Arithmeticae*. This was Gauss' favourite theorem and in his lifetime he gave eight different proofs. There are a great many proofs of the Law of Quadratic Reciprocity ranging from elementary to highly sophisticated, but none are straightforward. An elementary and understandable proof is given in [1], pages 133–135.

Using this law one can answer our earlier question of when 3 is a square root modulo p: it is when $p \equiv \pm 1 \pmod{12}$ — a key fact that we will use in our proof of Theorem 1. For completeness we give a proof as follows.

Proof. Let $p \geq 5$. Combining the Law of Quadratic Reciprocity with the observation that if $\gcd(a,p) = 1$ then $(a/p)^2 = 1$, we get

$$\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}\left(\frac{p}{3}\right) .$$

Therefore, $(3/p) = 1$ if and only if we have one of the following two conditions: (1) $\frac{p-1}{2}$ is even and p is a quadratic residue modulo 3; or (2) $\frac{p-1}{2}$ is odd and p is a quadratic nonresidue modulo 3.

Condition 1 is equivalent to the two congruences $p \equiv 1 \pmod 4$ and $p \equiv 1 \pmod 3$. By the Chinese Remainder Theorem this is equivalent to the single congruence $p \equiv 1 \pmod{12}$. Condition 2 is equivalent to the two congruences $p \equiv 3 \pmod 4$ and $p \equiv 2 \pmod 3$. By the Chinese Remainder Theorem this is equivalent to the single congruence $p \equiv 11 \pmod{12}$. Thus,

$$\left(\frac{3}{p}\right) = 1 \iff p \equiv \pm 1 \pmod{12} .$$

Hensel's lemma

Hensel's lemma answers the following type of problem. Suppose we are told that the congruence $x^2 \equiv 28 \pmod{37}$ has the solutions $x = 18$ and $x = 19$. What does this piece of information tell us about the existence of solutions to the congruence

$$x^2 \equiv 28 \left(\text{mod } 37^3\right) ?$$

Hensel's lemma gives an affirmative answer to this question.

Theorem 4 (Hensel's lemma, quadratic version) Let a be a quadratic residue modulo p. Then the congruence

$$x^2 \equiv a \left(\text{mod } p^k\right)$$

has a solution for all $k \geq 1$.

The basic idea of the proof is to start with a solution of $x^2 \equiv a \pmod p$ and "lift" it to $x^2 \equiv a \left(\text{mod } p^2\right)$ and repeat. This process is particularly straightforward for such quadratic congruences. For more general polynomial congruences $f(x) \equiv a \pmod{p^n}$, one needs the hypothesis that the solution is not a zero of the derivative $f'(x)$. See [1], Sections 4.3 and 7.5 for more details.

Finally, the way one solves quadratic congruences $x^2 \equiv a \pmod n$ for composite n is to first solve it for each prime divisor of n, and then combine Hensel's lemma with the Chinese Remainder Theorem to find a solution.

The Proof of Theorem 1

Suppose that $\mathcal{C}_n \cap \mathcal{H}_n \neq \emptyset$. Then $\mathcal{C}_p \cap \mathcal{H}_p \neq \emptyset$ for any prime divisor p of n. It is easy to check that both intersections $\mathcal{C}_2 \cap \mathcal{H}_2$ and $\mathcal{C}_3 \cap \mathcal{H}_3$ are empty, and consequently $p \geq 5$. We now prove that $p \equiv 1 \pmod{12}$. Let $(r, s) \in \mathcal{C}_p \cap \mathcal{H}_p$. We have

$$\begin{aligned} (r-s)^2 &\equiv -1 \pmod{p} , \\ (r+s)^2 &\equiv 3 \pmod{p} , \end{aligned}$$

that is, $(-1/p) = (3/p) = 1$, where $(\cdot/p)$ is the Legendre symbol. Since

$$\left(\frac{-1}{p}\right) = 1 \iff p \equiv 1 \pmod{4}$$

and

$$\left(\frac{3}{p}\right) = 1 \iff p \equiv \pm 1 \pmod{12} ,$$

we conclude that $p \equiv 1 \pmod{12}$.

Conversely let $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ be the canonical factorization of n and suppose $p_i \equiv 1 \pmod{12}$ for each i. We will show that $\mathcal{C}_n \cap \mathcal{H}_n \neq \emptyset$.

For each i both -1 and 3 are squares modulo p_i, since $p_i \equiv 1 \pmod{12}$. Using Theorem 4 we lift these square roots to the ${e_i}^{\text{th}}$ power, $p_i^{e_i}$. Let s_i and r_i be such that $s_i^2 \equiv -1 \pmod{p_i^{e_i}}$, and $r_i^2 \equiv 3 \pmod{p_i^{e_i}}$. Then

$$2^{-1} \cdot (r_i + s_i, r_i - s_i) \in \mathcal{C}_{p_i^{e_i}} \cap \mathcal{H}_{p_i^{e_i}} ,$$

where 2^{-1} denotes the inverse of 2 modulo $p_i^{e_i}$. We now invoke the Chinese Remainder Theorem to determine integers r and s such that

$$\begin{aligned} r &\equiv r_i \pmod{p_i^{e_i}} , \\ s &\equiv s_i \pmod{p_i^{e_i}} , \end{aligned}$$

for each $i = 1, 2, \ldots, t$. Clearly, $2^{-1} \cdot (r+s, r-s) \in \mathcal{C}_n \cap \mathcal{H}_n$, and the proof is complete.

References

[1] G.A. Jones and J.M. Jones, *Elementary Number Theory*, Springer-Verlag, 1998.

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PROBLEMS

Solutions to problems in this issue should arrive no later than ***1 May 2011****. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3576. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC be a triangle with interior points D, E, F such that $\angle FAB = \angle EAC$, $\angle FBA = \angle DBC$, $\angle DCB = \angle ECA$, $AF = AE$, $BF = BD$, and $CD = CE$. If R is the circumradius of ABC, r is the circumradius of EDF, and s is the semiperimeter of ABC, prove that the area of triangle EDF is $\frac{sr^2}{2R}$.

3577. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let H be the orthocentre of the acute triangle ABC with A' on the ray HA and such that $A'A = BC$. Define B', C' similarly. Prove that

$$\text{Area}(A'B'C') = 4\text{Area}(ABC) + \frac{a^2+b^2+c^2}{2}.$$

3578. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $a > 0$ and $b > 1$ be real numbers and let $f : [0,1] \to \mathbb{R}$ be a continuous function. Find

$$\lim_{n\to\infty} n^{a/b} \int_0^1 \frac{f(x)}{1+n^a x^b}\,dx\,.$$

3579. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $\alpha = \frac{\pi}{13}$ and

$$\begin{aligned} x_1 &= \tan(4\alpha) + 4\sin(\alpha) = -\tan(\alpha) + 4\sin(3\alpha)\,, \\ x_2 &= \tan(\alpha) + 4\sin(\alpha) = -\tan(4\alpha) + 4\sin(3\alpha)\,, \\ x_3 &= \tan(6\alpha) - 4\sin(6\alpha) = \tan(2\alpha) + 4\sin(5\alpha)\,. \end{aligned}$$

Prove that the length x_1 can be constructed with compass and straightedge and determine whether or not the same is true for x_2 and x_3.

3580. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k > 0$ and $m \geq 0$ be real numbers, and let $\{a\} = a - \lfloor a \rfloor$ denote the fractional part of a. Calculate

$$\int_0^1 \left\{ \frac{1}{x^k} - \frac{1}{(1-x)^k} \right\} x^m (1-x)^m \, dx \,.$$

3581. *Proposed by Zhi-min Song, Beizhen Middle School, Shandong Binzhou, China and Li Yin, Binzhou University, Shandong, China.*

Let a, b, c, d be positive reals with $0 < a \leq b \leq c \leq d$. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+d)} + \frac{1}{d(1+a)} \;\geq\; \frac{4}{\sqrt[4]{abcd}(1+\sqrt[4]{abcd})} \,.$$

3582. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let Γ_1, Γ_2 be circles of radius r with centres A, B (respectively), let $\{C, D\} = \Gamma_1 \cap \Gamma_2$, and suppose that $\angle BCA = 90^\circ$. A line through C intersects Γ_1 and Γ_2 again at E and F, respectively. The circle Γ with centre O and radius R passes through points E and F. A second line passes through C, is perpendicular to the segment EF, and intersects the circle Γ in G and H. Prove that $CH^2 + CG^2 = 4(R^2 - r^2)$.

3583. *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let α and β be nonnegative real numbers and define

$$\begin{aligned} a_n &= \big(n + \ln(n+1)\big) \prod_{k=1}^{n} \frac{\alpha + k + \ln k}{\beta + (k+1) + \ln(k+1)} \,, \\ p_n &= \big(\alpha + n + 1 + \ln(n+1)\big) \prod_{k=1}^{n} \frac{\alpha + k + \ln k}{\beta + (k+1) + \ln(k+1)} \,. \end{aligned}$$

Find those nonnegative real numbers α and β for which $\sum\limits_{n=1}^{\infty} a_n$ converges, and determine the relation between α and β that ensures that

$$\sum_{n=1}^{\infty} \left(a_n - p_n \ln\left(1 + \frac{1}{n+1}\right)\right) \;=\; (\alpha+1)(\alpha + 2 + \ln 2) - \frac{(\alpha+1)^2}{2} \,.$$

3584. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let ABC be a triangle with inradius r, side lengths a, b, c and medians m_a, m_b, m_c. Prove that $\frac{c}{m_a^2 m_b^2} + \frac{a}{m_b^2 m_c^2} + \frac{b}{m_c^2 m_a^2} \leq \frac{3\sqrt{3}}{27r^3}$.

3585. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $T_n(x)$ be the Chebyshev polynomial of the first kind defined by the recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 1$ and the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. Find all positive integers n such that

$$T_n(x) \leq (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}, \quad x \in [1,\infty).$$

3586. *Proposed by Shai Covo, Kiryat-Ono, Israel.*

For each positive integer n, a_n is the number of positive divisors of n of the form $4m+1$ minus the number of positive divisors of n of the form $4m+3$ (so $a_4 = 1$, $a_5 = 2$, and $a_6 = 0$). Evaluate the sum $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{a_n}{n}$.

3587★. *Proposed by Ignotus, Colegio Manablanca, Facatativá, Colombia.*

Define the *prime graph* of a set of positive integers as the graph obtained by letting the numbers be the vertices, two of which are joined by an edge if and only if their sum is prime.

(a) Prove that given any tree T on n vertices, there is a set of positive integers whose prime graph is isomorphic to T.

(b) For each positive integer n, determine $t(n)$, the smallest number such that for any tree T on n vertices, there is a set of n positive integers each not greater than $t(n)$ whose prime graph is isomorphic to T.

. . . .

3576. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit D, E et F trois points à l'intérieur d'un triangle ABC de sorte que $\angle FAB = \angle EAC$, $\angle FBA = \angle DBC$, $\angle DCB = \angle ECA$, $AF = AE$, $BF = BD$ et $CD = CE$. Si R et r sont respectivement les rayons du cercle exinscrit de ABC et de EDF, et s le demi-périmètre de ABC, montrer que l'aire du triangle EDF est $\frac{sr^2}{2R}$.

3577. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit H l'orthocentre du triangle acutangle ABC avec A' sur la demi-droite HA et tel que $A'A = BC$. On définit B' et C' de manière analogue. Montrer que

$$\text{Aire}(A'B'C') = 4\text{Aire}(ABC) + \frac{a^2+b^2+c^2}{2}.$$

3578. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $a > 0$ et $b > 1$ deux nombres réels et soit $f : [0,1] \to \mathbb{R}$ une fonction continue. Calculer

$$\lim_{n\to\infty} n^{a/b} \int_0^1 \frac{f(x)}{1+n^a x^b}\,dx\,.$$

3579. *Proposé par Peter Y. Woo, Université Biola, La Mirada, CA, É-U.*

Soit $\alpha = \frac{\pi}{13}$ et

$$\begin{aligned} x_1 &= \tan(4\alpha)+4\sin(\alpha) = -\tan(\alpha)+4\sin(3\alpha)\,,\\ x_2 &= \tan(\alpha)+4\sin(\alpha) = -\tan(4\alpha)+4\sin(3\alpha)\,,\\ x_3 &= \tan(6\alpha)-4\sin(6\alpha) = \tan(2\alpha)+4\sin(5\alpha)\,. \end{aligned}$$

Montrer que la longueur x_1 est constructible avec la règle et le compas, et décider si oui ou non il en va de même pour x_2 et x_3.

3580. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $k > 0$ et $m \geq 0$ deux nombres réels. On désigne par $\{a\} = a - \lfloor a \rfloor$ la partie fractionnaire de a. Calculer

$$\int_0^1 \left\{\frac{1}{x^k} - \frac{1}{(1-x)^k}\right\} x^m(1-x)^m\,dx\,.$$

3581. *Proposé par Zhi-min Song, École Secondaire Beizhen, Shandong Binzhou, Chine et Li Yin, Université Binzhou, Shandong, Chine.*

Soit a, b, c et d des nombres réels positifs tels que $0 < a \leq b \leq c \leq d$. Montrer que

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+d)}+\frac{1}{d(1+a)} \geq \frac{4}{\sqrt[4]{abcd}(1+\sqrt[4]{abcd})}\,.$$

3582. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Soit Γ_1 et Γ_2 deux cercles de rayon r et de centres respectifs A et B. Soit $\{C, D\} = \Gamma_1 \cap \Gamma_2$ et supposons que $\angle BCA = 90°$. Une droite par C coupe respectivement Γ_1 et Γ_2 en E et F. Le cercle Γ de centre O et de rayon R passe par les points E et F. Une seconde droite passe par C, est perpendiculaire au segment EF et coupe le cercle Γ en G et H. Montrer que $CH^2 + CG^2 = 4(R^2 - r^2)$.

3583. *Proposé par Paolo Perfetti, Département de Mathématiques, Université de Rome, "Tor Vergata", Rome, Italie.*

Soit α et β deux nombres réels non négatifs. On définit

$$a_n = \big(n + \ln(n+1)\big) \prod_{k=1}^{n} \frac{\alpha + k + \ln k}{\beta + (k+1) + \ln(k+1)},$$

$$p_n = \big(\alpha + n + 1 + \ln(n+1)\big) \prod_{k=1}^{n} \frac{\alpha + k + \ln k}{\beta + (k+1) + \ln(k+1)}.$$

Trouver pour quels nombres réels non négatifs α et β la série $\sum_{n=1}^{\infty} a_n$ converge et déterminer la relation entre α et β assurant que

$$\sum_{n=1}^{\infty} \left(a_n - p_n \ln\left(1 + \frac{1}{n+1}\right)\right) = (\alpha+1)(\alpha + 2 + \ln 2) - \frac{(\alpha+1)^2}{2}.$$

3584. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit respectivement r, a, b, c, m_a, m_b, m_c le rayon du cercle inscrit d'un triangle ABC, la longueur de ses côtés et celle de ses médianes. Montrer que $\frac{c}{m_a^2 m_b^2} + \frac{a}{m_b^2 m_c^2} + \frac{b}{m_c^2 m_a^2} \le \frac{3\sqrt{3}}{27r^3}$.

3585. *Proposé par Arkady Alt, San José, CA, É-U.*

Soit $T_n(x)$ le polynôme de Tchebychev du premier type défini par la récurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ pour $n \ge 1$ et les conditions initiales $T_0(x) = 1$ et $T_1(x) = x$. Trouver tous les entiers $n > 0$ tels que

$$T_n(x) \le (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}, \quad x \in [1, \infty).$$

3586. *Proposé par Shai Covo, Kiryat-Ono, Israël.*

Pour tout entier positif n, soit a_n le nombre des diviseurs positifs de n de la forme $4m+1$ moins le nombre de ceux ayant la forme $4m+3$ (p.ex. $a_4 = 1$, $a_5 = 2$ et $a_6 = 0$). Evaluer la somme $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{a_n}{n}$.

3587★. *Proposé par Ignotus, Colegio Manablanca, Facatativá, Colombie.*

On définit le *graphe premier* d'un ensemble d'entiers positifs comme le graphe obtenu en considérant les nombres comme des sommets, reliés deux à deux par une arête si et seulement si leur somme est un nombre premier.

(a) Montrer qu'étant donné un arbre quelconque T à n sommets, il existe un ensemble d'entiers positifs dont le graphe premier est isomorphe à T.

(b) Pour chaque entier positif n, trouver le plus petit nombre $t(n)$ tel que, pour tout arbre T à n sommets, il existe un ensemble de n entiers positifs pas plus grands que $t(n)$ et dont le graphe premier est isomorphe à T.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Our apologies for omitting correct solutions to problem 3466 by George Apostolopoulos, Messolonghi, Greece, and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

3475. [2009 : 463, 465] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be an equilateral triangle with side length a, and let P be a point on the line BC such that $AP = 2x > a$. Let M be the midpoint of AP. If $\frac{BM}{x} = \frac{BP}{a} = \alpha$ and $\frac{CM}{x} = \frac{CP}{a} = \beta$, find x, α, and β.

Solution by Joe Howard, Portales, NM, USA.

We show that $x = \frac{a}{\sqrt{2}}$, $\alpha = \frac{1+\sqrt{5}}{2}$, and $\beta = \frac{-1+\sqrt{5}}{2}$ without assuming that $\triangle ABC$ is equilateral; rather we assume that $AB = BC = a$, and that $AP = 2x > AC$ with C between B and P. The equations for α and β then imply that

$$\alpha \;=\; \frac{BP}{a} \;=\; \frac{a+CP}{a} \;=\; 1+\frac{CP}{a} \;=\; 1+\beta\,. \tag{1}$$

Stewart's theorem applied to $\triangle ABP$ and cevian BM yields

$$AB^2 \cdot PM + BP^2 \cdot MA \;=\; AP(BM^2 + PM \cdot MA)\,,$$

or

$$a^2x + a^2\alpha^2x \;=\; 2x(x^2\alpha^2 + x^2)\,,$$

whence,

$$a \;=\; \sqrt{2}x\,. \tag{2}$$

Stewart's theorem applied to $\triangle BMP$ and cevian MC yields

$$BM^2 \cdot CP + MP^2 \cdot BC \;=\; BP(CM^2 + BC \cdot CP)\,,$$

or

$$\alpha^2x^2 \cdot a\beta + x^2 \cdot a \;=\; a\alpha(\beta^2x^2 + a \cdot a\beta)\,.$$

From (2) this equation becomes $\alpha^2\beta + 1 = \alpha(\beta^2 + 2\beta)$, which upon using the relation in (1) becomes $(\beta^2 + 2\beta + 1)\beta + 1 = (\beta + 1)(\beta^2 + 2\beta)$, and reduces to

$$\beta^2 + \beta - 1 \;=\; 0\,.$$

Since $\beta > 0$, we conclude that $\beta = \dfrac{-1+\sqrt{5}}{2}$. Finally, using (1) again, we get that $\alpha = \dfrac{1+\sqrt{5}}{2}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2 solutions); MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTINEL MORTICI, Valahia University of Târgoviște, Romania; JOEL SCHLOSBERG, Bayside, NY, USA; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect submission.

Applying Stewart's theorem to $\triangle ACP$ with cevian CM, we deduce that $a = AC$, so that $\triangle ABC$ is necessarily equilateral and, consequently, the conditions stated in the problem are consistent. Although most of the submitted solutions were based on Stewart's theorem and the cosine law, a few exploited other notable features of the configuration such as the cyclic quadrilateral $ABCM$ and the similar triangles BMP and ACP.

Bataille observed that his result provides a simple way to construct line segments having lengths $\frac{1+\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2}$ given an equilateral triangle ABC with sides of unit length: construct a unit segment CQ perpendicular to AC at C; the circle with centre A and radius $AQ = \sqrt{2}$ intersects BC at P. The segments BP and CP have the desired lengths.

3476. [2009 : 463, 466] *Proposed by Michel Bataille, Rouen, France.*

Let ℓ be a line and O be a point not on ℓ. Find the locus of the vertices of the rectangular hyperbolas centred at O and tangent to ℓ. (A hyperbola is rectangular if its asymptotes are perpendicular.)

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Choose coordinates so that O is the origin and the equation of ℓ is $x = 1$. Let $\mathcal{R}$ denote the family of all rectangular hyperbolas with centre O that are tangent to ℓ, and let $A : (1, 0)$ be the foot of the perpendicular from O to ℓ. We will see that the desired locus is the hyperbola with the equation $x^2 - y^2 = 1$.

Lemma. The area of any triangle formed by the asymptotes of a rectangular hyperbola together with one of its tangents equals the square of the distance between the centre of the hyperbola and one of its vertices.

It is easier to provide a proof of this familiar result than to look up a reference: We introduce coordinates so that the rectangular hyperbola has equation $xy = a^2$; the tangent to this hyperbola at the point $(ak, a/k)$ is $y = -\dfrac{1}{k^2}x + \dfrac{2a}{k}$, which forms with the asymptotes $x = 0$, $y = 0$ a triangle whose vertices are $(0, 0)$, $(0, 2a/k)$ and $(2ak, 0)$, and whose area (for every k) is the constant $2a^2$. Since the vertices of $xy = a^2$ are (a, a) and $(-a, -a)$, the distance from the centre $(0, 0)$ to a vertex is $2a^2$.

Returning to the main problem, let V be the vertex of the branch for which $x > 0$ of any rectangular hyperbola of $\mathcal{R}$, and define $\theta = \angle AOV$. Because OV bisects the right angle formed by the two asymptotes, the area of the triangle these asymptotes form with the tangent ℓ is

$$OV^2 \;=\; \frac{1}{2}\sec(45^\circ - \theta)\sec(45^\circ + \theta) \;=\; \frac{1}{\cos 2\theta}\,.$$

Hence, V lies on the curve whose polar equation is $r^2 = \dfrac{1}{\cos 2\theta}$. We recognize this as the rectangular hyperbola $x^2 - y^2 = 1$ of $\mathcal{R}$ that has A as its vertex. Because $-45^\circ < \theta < 45^\circ$, it is clear that every point of this hyperbola is a vertex of some member of $\mathcal{R}$, whence the locus of vertices is the entire hyperbola.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; and the proposer.

3477. [2009 : 463, 466] *Proposed by Michel Bataille, Rouen, France.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2y^2\big(f(x+y) - f(x) - f(y)\big) \;=\; 3(x+y)f(x)f(y)$$

for all real numbers x and y.

Similar solutions by Edward J. Barbeau, University of Toronto, Toronto, ON and Salvatore Ingala, student, Scuola Superiore di Catania, University of Catania, Catania, Italy.

We prove that the only two solutions are the zero function, $f(x) = 0$, $x \in \mathbb{R}$, and the cube function $f(x) = x^3$, $x \in \mathbb{R}$. Clearly these two functions are solutions.

If f is the zero-function, then we are done, so henceforth we assume that f is not the zero function.

We first show that $f(0) = 0$.

Assume for the sake of contradiction that $f(0) \neq 0$. Setting $y = 0$ yields $0 = 3xf(x)f(0)$. Thus, $f(x) = 0$ for all $x \neq 0$. Then setting $x = 1$, $y = -1$ we obtain $f(0) = 0$, a contradiction.

Thus, $f(0) = 0$.

Now setting $y = -x$ yields

$$x^4[-f(x) - f(-x)] \;=\; 0\,,$$

and since $f(0) = 0$ we have $f(-x) = -f(x)$ for all x.

Setting $y = x$ for $x \neq 0$ yields

$$f(2x) \;=\; \frac{6}{x^3}f(x)^2 + 2f(x)\,.$$

Now picking $z \neq 0$ and setting $x = 2z$, $y = -z$ yields

$$4z^4\left[2f(z) - f(2z)\right] \;=\; 3zf(2z)f(-z) \;=\; -3zf(2z)f(z)\,.$$

Hence,

$$4z^3\left(\frac{6}{z^3}f(z)^2\right) \;=\; 3f(z)\left(\frac{6}{z^3}f(z)^2 + 2f(z)\right)\,.$$

Thus,

$$24z^3f(z)^2 \;=\; 18f(z)^3 + 6z^3f(z)^2\,,$$

or

$$z^3f(z)^2 \;=\; f(z)^3\,.$$

Therefore, for all $z \neq 0$, either $f(z) = 0$ or $f(z) = z^3$. We show that if f is not the zero function, then $f(z) = z^3$ for all z.

Suppose, for the sake of contradiction, that $f(z) = 0$ for some $z \neq 0$.

By putting $y = z$ in the original equation we get that

$$x^2z^2[f(x+z) - f(x)] \;=\; 0\,,$$

thus $f(x+z) = f(x)$ for all $x \neq 0$.

Since f is not the zero function, there exists an $x \neq 0$ so that $f(x) \neq 0$. Then

$$x^3 \;=\; f(x) \;=\; f(x+z)\,.$$

which contradicts the fact that $f(x+z) = 0$ or $f(x+z) = (x+z)^3$.

Thus, $f(z) \neq 0$ for all $z \neq 0$, and hence $f(z) = z^3$ for all $z \neq 0$.

This completes the proof.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incorrect solution and one incomplete solution were submitted.

3478. [2009 : 463, 466] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a and b be positive real numbers. Prove that

$$\frac{a}{b} \;+\; \frac{b}{a} \;+\; \sqrt{1 + \frac{2ab}{a^2+b^2}} \;\geq\; 2 + \sqrt{2}\,.$$

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let $x = \frac{a}{b}$, then

$$\frac{a}{b} + \frac{b}{a} + \sqrt{1 + \frac{2ab}{a^2+b^2}} \;=\; x + \frac{1}{x} + \frac{x+1}{\sqrt{x^2+1}}\,.$$

Let $f : (0,\infty) \to (0,\infty)$ be given by $f(x) = x + \frac{1}{x} + \frac{x+1}{\sqrt{x^2+1}}$. From the derivative

$$\begin{aligned} f'(x) &= 1 - \frac{1}{x^2} + \frac{1-x}{(x^2+1)^{3/2}} \\ &= (x-1)\left(\frac{x+1}{x^2} - \frac{1}{(x^2+1)^{3/2}}\right) \\ &= (x-1)\frac{(x+1)\left(x^2+1\right)^{3/2} - x^2}{x^2\left(x^2+1\right)^{3/2}} \\ &= (x-1)\frac{x^8+2x^7+4x^6+6x^5+5x^4+6x^3+4x^2+2x+1}{x^2\left(x^2+1\right)^{3/2}\left(x^2+(x+1)\left(x^2+1\right)^{3/2}\right)} \end{aligned}$$

we see that $f'(x) \le 0$ for $x \in (0,1]$ while $f'(x) \ge 0$ for $x \in [1,\infty)$. Therefore, $f(1) = 2+\sqrt{2}$ is the absolute minimum value of f on $(0,\infty)$. Equality occurs if and only if $\frac{a}{b} = 1$, that is, if and only if $a = b$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USAROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; AARON ESSNER, MARK FARRENBURG, and LUKE E. HARMON, students, Southeast Missouri State University, Cape Girardeau, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; DUNG NGUYEN MANH, Student, Hanoi University of Technology, Hanoi, Vietnam; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Tucson, AZ, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; PANOS E. TSAOUSSOGLOU, Athens, Greece; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; LUKE WESTBROOK, student, Southeast Missouri State University, Cape Girardeau, MO, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. One incomplete solution was submitted and two submitted solutions were disqualified.

3479. *Proposed by Jonathan Schneider, student, University of Toronto Schools, Toronto, ON.*

The real numbers x, y, and z satisfy the system of equations

$$\begin{aligned} x^2 - x &= yz + 1\,, \\ y^2 - y &= xz + 1\,, \\ z^2 - z &= xy + 1\,. \end{aligned}$$

Find all solutions (x, y, z) of the system and determine all possible values of $xy + yz + zx + x + y + z$ where (x, y, z) is a solution of the system.

Solution by George Apostolopoulos, Messolonghi, Greece.

By subtracting the second equation from the first one we obtain

$$\begin{aligned} & x^2 - y^2 - x + y \;=\; z(y - x) \\ \Longleftrightarrow\;\; & (x + y)(x - y) - (x - y) + z(x - y) \;=\; 0 \\ \Longleftrightarrow\;\; & (x - y)(x + y + z - 1) \;=\; 0 \\ \Longleftrightarrow\;\; & x \;=\; y \quad \text{or} \quad x + y + z \;=\; 1\,. \end{aligned}$$

Similarly, we deduce that $x = z$ or $x + y + z = 1$, and that $y = z$ or $x + y + z = 1$.

Thus, if $x + y + z \neq 1$, then $x = y = z$ and the first given equation yields that $x = -1$. By symmetry we deduce that

$$(x, y, z) \;=\; (-1, -1, -1) \tag{1}$$

is the only solution.

Otherwise $x+y+z = 1$, and we now classify the solutions in this case.

Substituting $z = 1-x-y$ into the first given equation and simplifying yields

$$x^2 + (y - 1)x + (y^2 - y - 1) \;=\; 0\,,$$

hence by the quadratic solution formula

$$x \;=\; \frac{(1 - y) \pm \sqrt{-3y^2 + 2y + 5}}{2}\,.$$

Since x is a real number, we must have $D = -3y^2+2y+5 \geq 0$, which holds if and only if $y \in [-1, \frac{5}{3}]$. For y in this range we obtain

$$z \;=\; 1 - x - y \;=\; \frac{(1 - y) \mp \sqrt{D}}{2}\,,$$

so that the solutions are

$$(x, y, z) \;=\; \left(\frac{(1 - y) \pm \sqrt{D}}{2}, y, \frac{(1 - y) \mp \sqrt{D}}{2}\right)\,, \quad y \in \left[-1, \frac{5}{3}\right]\,. \tag{2}$$

Finally, in either case (1) or (2), it is a straightforward calculation to show that $xy + yz + zx + x + y + z = 0$.

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; EDWARD J. BARBEAU, University of Toronto, Toronto, ON; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; BAO CHANGJIN, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CALVIN DENG, student, William G. Enloe High School, Cary, North Carolina, USA; OLIVER GEUPEL, Brühl, NRW, Germany; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. One incorrect solution and four incomplete solutions were submitted.

3480. *Proposed by Bianca-Teodora Iordache, Carol I National College, Craiova, Romania.*

Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n \ \geq\ a_1 a_2 \cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Prove that $a_1^n + a_2^n + \cdots + a_n^n \geq a_1^{n-1} a_2^{n-1} \cdots a_n^{n-1}$. Find a necessary and sufficient condition for equality to hold.

Comments by Oliver Geupel, Brühl, NRW, Germany and Albert Stadler, Herrliberg, Switzerland.

Oliver Geupel comments that there is a duplication of problem 3480 with problem 880 in the *College Mathematical Journal*, 2008, May issue.

Albert Stadler comments that problem 3480 has already been published by the same author as Aufgabe 1251 in *Elemente der Mathematick*, 2008, issue 1. The solution appeared in the same journal in 2009, issue 1.

Solutions were received from ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; DUNG NGUYEN MANH, Student, Hanoi University of Technology, Hanoi, Vietnam; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3481. [2009 : 464, 466] *Proposed by Joe Howard, Portales, NM, USA.*

Let $\triangle ABC$ have at most one angle exceeding $\frac{\pi}{3}$. If $\triangle ABC$ has area F and side lengths a, b, and c, prove that

$$(ab + bc + ca)^2 \ \geq\ 4\sqrt{3} \cdot F\left(a^2 + b^2 + c^2\right).$$

Composite of similar solutions by Michel Bataille, Rouen, France and Oliver Geupel, Brühl, NRW, Germany.

The given inequality actually holds for all triangles. Recall first the Hadwiger-Finsler Inequality

$$a^2+b^2+c^2 \;\geq\; 4\sqrt{3}F+(a-b)^2+(b-c)^2+(c-a)^2\,, \tag{1}$$

two proofs of which are given in *Problem-Solving Strategies* by A. Engel, Springer, 1998; pp. 173-4. We rewrite (1) as

$$2(ab+bc+ca)-(a^2+b^2+c^2) \;\geq\; 4\sqrt{3}F\,. \tag{2}$$

Note that

$$\begin{aligned}
&(ab+bc+ca)^2\\
&= \Big((a^2+b^2+c^2)-(ab+bc+ca)\Big)^2\\
&\qquad -(a^2+b^2+c^2)^2+2(ab+bc+ca)(a^2+b^2+c^2)\\
&= \frac{1}{4}\Big((a-b)^2+(b-c)^2+(c-a)^2\Big)^2\\
&\qquad +\Big(2(ab+bc+ca)-(a^2+b^2+c^2)\Big)(a^2+b^2+c^2)\\
&\geq \Big(2(ab+bc+ca)-(a^2+b^2+c^2)\Big)(a^2+b^2+c^2)\,. \qquad (3)
\end{aligned}$$

The result now follows from (2) and (3).

It is easy to see that equality holds if and only if $a=b=c$; that is, if and only if $\triangle ABC$ is equilateral.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; KEE-WAI LAU, Hong Kong, China; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3482. [2009 : 464, 467] *Proposed by José Luis Díaz-Barrero and Josep Rubió-Masseguí, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $a_n \neq 0$ and $p(z) = \sum\limits_{k=0}^{n} a_k z^k$ be a polynomial with complex coefficients and zeros $z_1, z_2, \ldots, z_n$, such that $|z_k| < R$ for each k. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{R^2-|z_k|^2}} \;\geq\; \frac{2}{R^2}\left|\frac{a_{n-1}}{a_n}\right|\,.$$

When does equality occur?

Solution by Michel Bataille, Rouen, France.

For $x \in (0, R^2)$, we have $x(R^2 - x) \le \left(\frac{x + R^2 - x}{2}\right)^2$ (by the AM-GM Inequality). Hence,

$$\frac{1}{\sqrt{R^2 - x}} \;\ge\; \frac{2}{R^2}\sqrt{x}\,,$$

which still holds if $x = 0$. Note that equality holds if and only if $x = R^2 - x$, that is, $x = \frac{R^2}{2}$. It follows that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{R^2 - |z_k|^2}} \;\ge\; \frac{2}{R^2}\sum_{k=1}^{n} |z_k|\,. \tag{1}$$

By the triangle inequality,

$$\left|\sum_{k=1}^{n} z_k\right| \;\le\; \sum_{k=1}^{n} |z_k| \tag{2}$$

and $\sum\limits_{k=1}^{n} z_k = \frac{a_{n-1}}{a_n}$ (Vieta's formula), hence

$$\sum_{k=1}^{n} \frac{1}{\sqrt{R^2 - |z_k|^2}} \;\ge\; \frac{2}{R^2}\left|\frac{a_{n-1}}{a_n}\right|\,, \tag{3}$$

as required.

Equality holds in (1) if and only if $|z_k| = \frac{R}{\sqrt{2}}$ for $k = 1, 2, \ldots, n$ and in (2) if the nonzero z_k's have the same argument. As a result, equality in (3) is equivalent to $z_1 = z_2 = \cdots = z_n = \frac{R}{\sqrt{2}} \cdot e^{i\theta}$ for some $\theta \in [0, 2\pi)$ that is, when $p(z)$ is of the form $a_n\left(z - \frac{R}{\sqrt{2}} \cdot e^{i\theta}\right)^n$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposers. Three incomplete solutions were submitted.

3483. [2009 : 464, 467] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let $n \ge 3$ be an integer and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\left(\frac{x_1}{x_2}\right)^{n-2} + \left(\frac{x_2}{x_3}\right)^{n-2} + \cdots + \left(\frac{x_n}{x_1}\right)^{n-2} \;\ge\; \frac{x_1 + x_2 + \cdots + x_n}{\sqrt[n]{x_1 x_2 \cdots x_n}}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let

$$N = \frac{(n-3)n}{2} + \sum_{k=1}^{n-1} k = (n-2)n\,.$$

The arithmetic mean of the following N numbers

$$\underbrace{1,1,1,\ldots,1}_{\frac{(n-3)n}{2}\text{ numbers}}\,,\ \underbrace{\left(\frac{x_1}{x_2}\right)^{n-2},\ldots,\left(\frac{x_1}{x_2}\right)^{n-2}}_{n-1\text{ numbers}}\,,$$

$$\underbrace{\left(\frac{x_2}{x_3}\right)^{n-2},\ldots,\left(\frac{x_2}{x_3}\right)^{n-2}}_{n-2\text{ numbers}}\,,\ \ldots\,,\ \underbrace{\left(\frac{x_{n-1}}{x_n}\right)^{n-2}}_{\text{one number}}$$

is not less than their geometric mean; that is

$$\frac{(n-3)n}{2} + \sum_{k=1}^{n-1}(n-k)\left(\frac{x_k}{x_{k+1}}\right)^{n-2}$$

$$\geq N\left[\frac{x_1^{(n-2)(n-1)}}{\prod\limits_{k=2}^{n} x_k^{n-2}}\right]^{1/N} = \frac{Nx_1}{\sqrt[n]{x_1x_2\cdots x_n}}\,.$$

We obtain n variants of this inequality by cyclic shifts of the index k. Adding these n inequalities and using the AM–GM Inequality again yields

$$\begin{aligned}
&\frac{(n-3)n^2}{2} + \frac{(n-1)n}{2}\left[\left(\frac{x_1}{x_2}\right)^{n-2} + \left(\frac{x_2}{x_3}\right)^{n-2} + \cdots + \left(\frac{x_n}{x_1}\right)^{n-2}\right]\\
&\geq\ N\left(\frac{x_1+x_2+\cdots+x_n}{\sqrt[n]{x_1x_2\cdots x_n}}\right)\\
&\geq\ \frac{(n-3)n}{2}\cdot\frac{n\sqrt[n]{x_1x_2\cdots x_n}}{\sqrt[n]{x_1x_2\cdots x_n}} + \frac{(n-1)n}{2}\cdot\frac{x_1+x_2+\cdots+x_n}{\sqrt[n]{x_1x_2\cdots x_n}}\\
&=\ \frac{(n-3)n^2}{2} + \frac{(n-1)n}{2}\cdot\frac{x_1+x_2+\cdots+x_n}{\sqrt[n]{x_1x_2\cdots x_n}}\,.
\end{aligned}$$

The desired inequality follows immediately.

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; JOE HOWARD, Portales, NM, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3484★. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let N be a positive integer with decimal expansion $N = a_1a_2\dots a_r$, where r is the number of decimal digits and $0 \le a_i \le 9$ for each i, except for a_1, which must be positive. Let $s(N) = a_1 + a_2 + \cdots + a_r$. Find all pairs (N, p) of positive integers such that $\big(s(N)\big)^p = s\big(N^p\big)$.

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

We relabel the decimal expansion as $N = a_{r-1}a_{r-2}\dots a_1a_0$, so that $N = a_{r-1}10^{r-1} + a_{r-2}10^{r-2} + \cdots + a_1 10^1 + a_0 10^0$. Then for any positive integer p we have

$$N^p = \sum_{n=0}^{p(r-1)} q_n 10^n ,$$

where

$$q_n = \sum_{\substack{(k_1,\dots,k_p)\in\{1,\dots,r\}^p \\ k_1+\cdots+k_p=n}} a_{k_1}a_{k_2}\cdots a_{k_p} .$$

We call the number N *carry-free of order* p, or CF-p for short, if all the numbers q_0, q_1, ..., $q_{p(r-1)}$ are less than 10. This means that no carries occur in the $p-1$ multiplications needed to compute N^p.

If for positive integers $M = \sum b_k 10^k$ and $N = \sum c_j 10^j$ the long multiplication of MN is free of carries, then we have $s(MN) = s(M)s(N)$, as is verified by considering the polynomial multiplication $\left(\sum b_k x^k\right)\left(\sum c_j x^j\right)$ and substituting $x = 10$. On the other hand, the digit sum decreases by 9 each time a carry occurs.

Therefore, a pair (N, p) is a solution if and only if N is CF-p.

We observe that if N is CF-$(p+1)$, then N is CF-p, and furthermore decreasing any (positive) digit of a CF-p integer leaves a CF-p integer.

Proposition 1 For $p \ge 5$, the number N is CF-p if and only if $N = 10^{r-1}$.

Proof: Clearly, $N = 10^{r-1}$ is CF-p.

Conversely, assume that N is CF-p. Then N is CF-5 with each digit either zero or one. If N contains the digit one at least twice, then zeros can be eliminated so that $a_0 = a_{r-1} = 1$, and we obtain

$$\begin{aligned} q_{2r-2} &= \left|\{(k_1,\dots,k_5)\in\{1,\dots,r\}^5 \mid k_1+\cdots+k_5 = 2r+3\}\right| \\ &\ge \binom{5}{2} a_0^3 a_{r-1}^2 = 10 , \end{aligned}$$

a contradiction. ■

Proposition 2 The number N is CF-4 if and only if it is CF-5 or $N = 10^k + 10^\ell$ where k and ℓ are distinct nonnegative integers.

Proof: The number $N = 10^k + 10^\ell$ is CF-4, since $q_n \leq \max\limits_m \binom{4}{m} = 6$ for each n.

Conversely, let N be CF-4. Then each digit of N is either zero or one. Assume that the digit one occurs at least thrice, say $a_0 = a_t = a_{r-1} = 1$, $1 < t < r-1$. Then, $q_{2r+t-2} \geq \binom{4}{2}\binom{2}{1}a_0 a_t a_{r-1}^2 = 12$, a contradiction. ∎

Proposition 3 If N is CF-3, then $a_i \in \{0,\,1,\,2\}$ for each digit a_i of N, and N contains the digit 2 if and only if $N = 2 \cdot 10^{r-1}$.

Proof: Since $3^3 > 9$, it follows that $a_i \in \{0,\,1,\,2\}$ for each i.

Clearly, $N = 2 \cdot 10^{r-1}$ is CF-3.

Conversely, assume that the digit 2 occurs in the CF-3 number N with at least one more digit of N being nonzero.

Then a number of the form $N_1 = 2 \cdot 10^{r-1} + 1$ would also be CF-3. However, for N_1 we have $q_{2r-2} = \binom{3}{2} \cdot 2 \cdot 2 \cdot 1 = 12$, a contradiction. ∎

Proposition 4 If N is CF-2, then $a_i \in \{0,\,1,\,2,\,3\}$ for each digit a_i of N, and if N contains the digit 3, then all other digits of N are zero or one.

Proof: Since $4^2 > 9$, we have $a_i \in \{0,\,1,\,2,\,3\}$ for each i. The second statement follows from the fact that if $N = 3 \cdot 10^{r-1} + 2$, then $q_{r-1} = 12$. ∎

Clearly, any positive integer N is CF-1, and this observation completes our characterization.

One hankers for a more explicit description of the solutions when $p = 3$ and N consists of 0's and 1's, or when $p = 2$. In each of these cases Geupel exhibited infinitely many positive integers N that are solutions, and infinitely many that are not.

Richard I. Hess, Rancho Palos Verdes, CA, USA determined all solutions (N, p) with N having at most 5 digits.

The proposer indicated that the special case when $p = 2$ and N is a two-digit integer had been posed by him earlier in the October 2007 issue of the journal School Science Mathematics.

3485. [2009 : 465, 465] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let x, y, z be positive real numbers in the interval $[0,1]$. Prove that

$$\frac{x}{y+z+1} + \frac{y}{x+z+1} + \frac{z}{x+y+1} + (1-x)(1-y)(1-z) \;\leq\; 1\,.$$

Comment by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

This is Problem 5 from the USA Mathematical Olympiad, 1980. A solution can be found here: `http://www.artofproblemsolving.com/Resources/Papers/MildorfInequalities.pdf`.

Solutions were received from ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of

Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; ALBERT STADLER, Herrliberg, Switzerland; PANOS E. TSAOUSSOGLOU, Athens, Greece; TITU ZVONARU, Comănești, Romania; and the proposer. One incomplete solution was submitted.

Arslanagic; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Howard also noted that the problem is the same as Problem 5 of the USAMO.

3486. [2009 : 465, 467] *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b, and c be positive real numbers. Prove that

$$\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab} \;\le\; \frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\,.$$

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

The inequality to be proved is

$$\sum_{\text{cyclic}}\left(1-\frac{a^2}{a^2+bc}\right) \;\le\; \frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\,,$$

or

$$\frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}+\sum_{\text{cyclic}}\frac{a^2}{a^2+bc} \;\ge\; 3\,.$$

Applying the Cauchy–Schwarz inequality, $\|u\|^2\,\|v\|^2 \ge (u\cdot v)^2$, to

$$u \;=\; \left(\frac{a}{\sqrt{a^2+bc}},\frac{b}{\sqrt{b^2+ca}},\frac{c}{\sqrt{c^2+ab}}\right)\,,$$

$$v \;=\; \left(\sqrt{a^2+bc},\sqrt{b^2+ca},\sqrt{c^2+ab}\right)\,,$$

we deduce that

$$\sum_{\text{cyclic}}\frac{a^2}{a^2+bc} \;\ge\; \frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca} \;=\; \frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)}\,.$$

Hence, it suffices to show that

$$\frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} + \frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)} \;\ge\; 3\,. \qquad (1)$$

Since the inequality is homogeneous, we may assume that $a+b+c=1$. Then $(a,b,c)\cdot(b,c,a) \leq a^2+b^2+c^2 = 1-2(ab+ac+bc)$, so $3(ab+ac+bc) \leq 1$. Thus, there is some $0 \leq x \leq 1$ such that $ab+bc+ca = \dfrac{1-x^2}{3}$. Now, it has been shown (see *Mathematical Reflections*, issue 2, 2007, "On a class of three–variable inequalities", by Vo Quoc Ba Can) that

$$abc \ \leq \ \frac{(1-x)^2(1+2x)}{27}\,,$$

and thus, (1) follows from

$$\frac{1}{2+x^2} \ + \ \frac{1}{2}\sqrt[3]{\frac{1+x}{(1-x)(1+2x)}} \ \geq \ 1\,,$$

which is equivalent to

$$\frac{1}{8}\frac{1+x}{(1-x)(1+2x)} \ - \ \left(1-\frac{1}{2+x^2}\right)^3 \ \geq \ 0\,. \tag{2}$$

After some algebra, we see that (2) will follow from

$$x^2(16x^6-7x^5+41x^4-18x^3+30x^2-12x+4) \ \geq \ 0\,.$$

Evidently this last inequality holds, since it is the same as

$$x^2\left[16x^6+(41x^4-7x^5)+(18x^2-18x^3)+4(3x^2-3x+1)\right] \ \geq \ 0\,,$$

the quadratic $3x^2-3x+1$ takes only positive values, and $0 \leq x \leq 1$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incorrect solution was submitted.

3487★. [2009 : 465, 467] *Proposed by Neven Jurič, Zagreb, Croatia.*

Does the following hold for every positive integer n?

$$\sum_{k=0}^{n-1}(-1)^k\frac{1}{2n-2k-1}\binom{2n-1}{k} \ = \ (-1)^{n-1}\frac{16^n}{8n\binom{2n}{n}}\,.$$

Solution by George Apostolopoulos, Messolonghi, Greece, modified and expanded by the editor.

The answer is Yes. We start off by setting $l = 2n-1-k$. Then $2n-2k-1 = 2n-2(2n-1-l)-1 = -(2n-2l-1)$ and l takes the values $2n-1$, $2n-2$, $\dots$, n as k takes the values 0, 1, 2, $\dots$, $n-1$; respectively.

Hence,

$$\begin{aligned}(-1)^k\frac{1}{2n-2k-1}\binom{2n-1}{k} &= (-1)^{2n-1-l}\frac{1}{-(2n-2l-1)}\binom{2n-1}{l}\\ &= (-1)^l\frac{1}{2n-2l-1}\binom{2n-1}{l}.\end{aligned}$$

Thus, if S denotes the sum on the left side of the given identity, then we deduce that

$$\begin{aligned}2S &= \sum_{k=0}^{n-1}(-1)^k\frac{1}{2n-2k-1}\binom{2n-1}{k} + \sum_{l=n}^{2n-1}(-1)^l\frac{1}{2n-2l-1}\binom{2n-1}{l};\\ S &= \frac{1}{2}\sum_{k=0}^{2n-1}(-1)^k\frac{1}{2n-2k-1}\binom{2n-1}{k}\\ &= \frac{1}{2(2n-1)}\sum_{k=0}^{2n-1}(-1)^k\frac{\frac{1}{2}-n}{\frac{1}{2}-n+k}\binom{2n-1}{k}. \qquad (1)\end{aligned}$$

We now prove a lemma.

Lemma For any fixed non-integer constant α,

$$\sum_{k=0}^{n}(-1)^k\frac{\alpha}{\alpha+k}\binom{n}{k} = \frac{n!}{\prod_{i=1}^{n}(\alpha+i)}. \qquad (2)$$

Proof: Let A_n denote the left side of the identity to be proved.

Then,

$$\begin{aligned}A_n &= \sum_{k=0}^{n}(-1)^k\frac{\alpha}{\alpha+k}\binom{n}{k}\\ &= 1+\sum_{k=1}^{n-1}(-1)^k\frac{\alpha}{\alpha+k}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)+(-1)^n\frac{\alpha}{\alpha+n}\\ &= A_{n-1}+\sum_{k=1}^{n}(-1)^k\frac{\alpha}{\alpha+k}\binom{n-1}{k-1}\\ &= A_{n-1}+\frac{\alpha}{n}\sum_{k=0}^{n}(-1)^k\frac{k}{\alpha+k}\binom{n}{k}\\ &= A_{n-1}+\frac{\alpha}{n}\left(\sum_{k=0}^{n}(-1)^k\left(1-\frac{\alpha}{\alpha+k}\right)\binom{n}{k}\right)\\ &= A_{n-1}+\frac{\alpha}{n}\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}-\sum_{k=0}^{n}(-1)^k\frac{\alpha}{\alpha+k}\binom{n}{k}\right)\\ &= A_{n-1}-\left(\frac{\alpha}{n}\right)A_n,\end{aligned}$$

since $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (1-1)^n = 0$. Hence, we obtain the recurrence relation

$$A_n = \left(\frac{n}{\alpha+n}\right) A_{n-1}\,. \tag{3}$$

Now $A_1 = 1 - \dfrac{\alpha}{\alpha+1} = \dfrac{1}{\alpha+1}$, so the recurrence (3) yields (2). ■

Before continuing our proof, we follow the usual practice and use the notation $(2n-1)!!$ for the product $(2n-1)(2n-3)\cdots 3\cdot 1$ and $(2n-2)!!$ for the product $(2n-2)(2n-4)\cdots 4\cdot 2$.

It is easily seen that $\dfrac{(2n-2)!}{(2n-3)!!} = (2n-2)!! = 2^{n-1}(n-1)!$ and $(2n-1)!! = \dfrac{(2n)!}{2^n n!}$.

To complete our proof we take $\alpha = \dfrac{1}{2} - n$ and replace n by $2n-1$ in the lemma. Then we have

$$\begin{aligned}
&\sum_{k=0}^{2n-1}(-1)^k \frac{\frac{1}{2}-n}{k+\frac{1}{2}-n}\binom{2n-1}{k} \\
&= \frac{(2n-1)!}{\prod_{i=1}^{2n-1}\left(\frac{1}{2}-n+i\right)} = \frac{(2n-1)!}{\left(\frac{3}{2}-n\right)\left(\frac{5}{2}-n\right)\cdots\left(\frac{4n-1}{2}-n\right)} \\
&= \frac{(-1)^{2n-1}2^{2n-1}(2n-1)!}{(2n-3)!!(-1)(-3)\cdots(-(2n-1))} \\
&= \frac{(-1)^{n-1}2^{2n-1}(2n-1)!}{(2n-3)!!(2n-1)!!}\,.
\end{aligned} \tag{4}$$

From (1) and (3) we then have

$$\begin{aligned}
S &= \frac{(-1)^{n-1}2^{2n-1}(2n-1)!}{2(2n-1)(2n-1)!!(2n-3)!!} = \frac{(-1)^{n-1}2^{2n-2}(2n-2)!}{(2n-1)!!(2n-3)!!} \\
&= \frac{(-1)^{n-1}2^{2n-2}(2n-2)!!}{(2n-1)!!} = \frac{(-1)^{n-1}2^{3n-3}(n-1)!}{(2n-1)!!} \\
&= \frac{(-1)^{n-1}2^{3n-3}n!n!}{n(2n-1)!!n!} = \frac{(-1)^{n-1}2^{4n-3}(n!)^2}{n(2n)!} \\
&= \frac{(-1)^{n-1}16^n}{8n\binom{2n}{n}}\,,
\end{aligned}$$

and our proof is complete.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and ALBERT STADLER, Herrliberg, Switzerland.

Stan Wagon gave the usual comment that the proposed identity can by verified by using Mathematica *if one expresses the given sum in terms of the Pochhammer function. Curtis used Gauss' Hypergeometric function and the Gamma function, while Stadler used complex contour integration, the Residue Theorem, and the Gamma function as well. Our featured solution is the only elementary one submitted, though by no means easy.*