M386. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

Determine all real numbers x for which

$$\sqrt{2 + 4x - 2x^2} + \sqrt{6 + 6x - 3x^2} = x^2 - 2x + 6\,.$$

M387. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Temperature can be measured in degrees Fahrenheit (F) or in degrees Celsius (C). The two scales are related by the formula $F = 1.8C + 32$. When a two-digit integer degree temperature in Celsius is converted to Fahrenheit and rounded to the nearest integer degree, it turns out the ones and tens digits of the original Celsius temperature C sometimes switch places to give the rounded Fahrenheit equivalent. Find all two-digit integer values of C for which this occurs.

Mayhem Solutions

M344. *Proposed by the Mayhem Staff.*

Consider the square array

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

formed by listing the numbers 1 to 9 in order in consecutive rows. The sum of the integers on each diagonal is 15. If a similar array is constructed using the integers 1 to $10\,000$, what is the sum of the numbers on each diagonal?

Solution by José Hernández Santiago, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico, modified by the editor.

Let S_1 represent the sum of the integers on the diagonal that runs from top left to bottom right. Let S_2 represent the sum of the integers on the diagonal that runs from top right to bottom left.

Our array is 100 by 100. When we move one column to the right in the same row, the number increases by 1; when we move one column to the left in the same row, the number decreases by 1. When we move one row down in the same column, the number increases by 100.

On the top left to bottom right diagonal, each number is one column to the right and one row down, so is $1 + 100 = 101$ greater than the previous number on the diagonal. Thus,

$$\begin{aligned} S_1 &= 1 + 102 + 203 + \cdots + 9899 + 10000 \\ &= (100 \cdot 0 + 1) + (100 \cdot 1 + 2) + \cdots + (100 \cdot 98 + 99) \\ &\quad + (100 \cdot 99 + 100) \\ &= 100(0 + 1 + \cdots + 98 + 99) \ + \ (1 + 2 + \cdots + 99 + 100) \\ &= 100\left(\tfrac{1}{2}(99)(100)\right) \ + \ \tfrac{1}{2}(100)(101) \\ &= 100(99)(50) + 50(101) \ = \ 495000 + 5050 \ = \ 500050\,. \end{aligned}$$

On the top right to bottom left diagonal, each number is one column to the left and one row down, so is $-1 + 100 = 99$ greater than the previous number on the diagonal. Thus,

$$\begin{aligned} S_2 &= 100 + 199 + 298 + \cdots + 9802 + 9901 \\ &= (100(0) + 100) + (100(1) + 99) + \cdots + (100(98) + 2) \\ &\quad + (100(99) + 1) \\ &= 100(0 + 1 + \cdots + 98 + 99) \ + \ (100 + 99 + \cdots + 2 + 1)\,. \end{aligned}$$

Therefore, $S_2 = S_1 = 50050$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M345. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The area of isosceles $\triangle ABC$ is $q\sqrt{15}$. Given that $AB = 2BC$, express the perimeter of $\triangle ABC$ in terms of q.

Solution by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.

Let $BC = x$ and $AB = 2x$. Since $\triangle ABC$ is isosceles, then $AC = BC$ or $AC = AB$. Since $\triangle ABC$ is presumably not a degenerate triangle, we take $AC = AB = 2x$ (otherwise $AC = BC = x$ and $AB = 2x$, which would mean that the triangle was a straight line segment of length $2x$).

Let D be the midpoint of BC. Since $\triangle ABC$ is isosceles, then AD is perpendicular to BC. The length of this altitude is thus

$$\begin{aligned} AD &= \sqrt{AB^2 - \left(\tfrac{1}{2}BC\right)^2} \ = \ \sqrt{(2x)^2 - \left(\tfrac{1}{2}x\right)^2} \\ &= \sqrt{4x^2 - \frac{1}{4}x^2} \ = \ \sqrt{\frac{15}{4}x^2} \ = \ \frac{\sqrt{15}}{2}x\,. \end{aligned}$$

The area of $\triangle ABC$ is $q\sqrt{15}$ and also equal to $\frac{1}{2}(BC)(AD) = \frac{1}{2}x\left(\frac{\sqrt{15}}{2}x\right)$, so that $q\sqrt{15} = \frac{\sqrt{15}}{4}x^2$, or $x^2 = 4q$ and hence $x = 2\sqrt{q}$.

Thus, $AB + BC + CA = 2x + x + 2x = 5x = 10\sqrt{q}$ is the perimeter of the triangle.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were 2 incorrect solutions submitted.

M346. *Proposed by the Mayhem Staff.*

Without using a calculator, find the number of digits in the integer 2^{80}.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.

We determine the positive integer n for which $10^n \leq 2^{80} < 10^{n+1}$. This will imply that the integer 2^{80} has $n + 1$ digits.

Since $1000 = 10^3 < 2^{10} = 1024$, raising both sides to the exponent 8 we obtain $10^{24} < 2^{80}$.

Also, $65536 = 2^{16} < 10^5 = 100000$, hence $2^{80} < 10^{25}$ upon raising both sides to the exponent 5.

Thus, $10^{24} < 2^{80} < 10^{25}$, whence 2^{80} has 25 digits.

Also solved by RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were 5 incomplete solutions submitted.

Singh submitted a solution that included more of a thought process that led to the same outcome as that of Ajanovic.

M347. *Proposed by the Mayhem Staff.*

Four positive integers a, b, c, and d are such that

$$\begin{aligned}(a+b+c)d &= 420\,,\\ (a+c+d)b &= 403\,,\\ (a+b+d)c &= 363\,,\\ (b+c+d)a &= 228\,.\end{aligned}$$

Find the four integers.

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

First, we write out the prime factorizations of the right sides of the equations:

$$\begin{aligned}420 &= 2^2\cdot 3\cdot 5\cdot 7\,, & 403 &= 13\cdot 31\,,\\ 363 &= 3\cdot 11^2\,, & 228 &= 2^2\cdot 3\cdot 19\,.\end{aligned}$$

Next, we subtract the second equation from the first equation to obtain $(a+c)d + bd - (a+c)b - bd = 420 - 403$, or $(a+c)(d-b) = 17$. Since a, b, c, and d are positive integers, then $a+c > 1$. Since $(a+c)$ is a divisor of 17, which is a prime number, then $a+c = 17$ and so $d-b = 1$.

From the third equation, c is odd since it is a divisor of 363. Since $a+c = 17$, then $a = 17 - c$ so a is even. Also, $1 \le c \le 15$ and $2 \le a \le 16$.

Since a is an even divisor of $228 = 2^2 \cdot 3 \cdot 19$ between 2 and 16 inclusive, then a could be 2, 4, 6, or 12. Since c is an odd divisor of $363 = 3 \cdot 11^2$ between 1 and 15 inclusive, then c could be 1, 3, or 11. Since $a + c = 17$, then a must equal 6 and c must equal 11.

So we know that $a = 6$, $c = 11$, and $d = b + 1$. Substituting these into the second equation, we obtain

$$\begin{aligned}(6+11+b+1)b &= 403\,;\\ (b+18)b &= 403\,;\\ b^2+18b-403 &= 0\,;\\ (b-13)(b+31) &= 0\,.\end{aligned}$$

Since b is a positive integer, we have $b = 13$, whence $d = 14$.

Therefore, $(a,b,c,d) = (6,13,11,14)$, which we can check satisfies each of the four equations.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were 2 incomplete solutions and 1 incorrect solution submitted.

M348. *Proposed by the Mayhem Staff.*

The perimeter of a sector of a circle is 12 (the perimeter includes the two radii and the arc). Determine the radius of the circle that maximizes the area of the sector.

Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia, modified by the editor.

Let r be the radius of the circle, let θ be the central angle of the sector measured in radians, and let A be the area of the sector.

The length of the arc of the sector is $\frac{\theta}{2\pi}(2\pi r) = \theta r$. Since the perimeter of the sector is 12, then the arc length of the sector also equals $12 - 2r$ (by subtracting the total length of the two radii). Therefore, $\theta r = 12 - 2r$.

Also, the area of the sector is $A = \frac{\theta}{2\pi}\left(\pi r^2\right) = \frac{1}{2}\theta r^2$. Thus, we have $A = \frac{1}{2}(\theta r)r = \frac{1}{2}(12-2r)r = 6r - r^2$.

The vertex of the parabola defined by this expression corresponds to the maximum value for A. This occurs when $r = -\frac{6}{2(-1)} = 3$. Therefore, the radius of the circle that maximizes the area of the sector is $r = 3$ (which gives a maximum area of $A = 9$).

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M349. *Proposed by the Mayhem Staff.*

(a) Find all ordered pairs of integers (x, y) with $\frac{1}{x} + \frac{1}{y} = \frac{1}{5}$.

(b) How many ordered pairs of integers (x, y) are there with

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}?$$

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

First, in (a) note that x cannot equal 0 or 5, and y cannot equal 0 or 5, since neither the denominators nor the individual fractions can equal 0. Solving the equation for y, we find that $\frac{1}{y} = \frac{1}{5} - \frac{1}{x} = \frac{x-5}{5x}$ or

$$y = \frac{5x}{x-5} = \frac{(5x-25)+25}{x-5} = 5 + \frac{25}{x-5}.$$

Since y is an integer, then so is $\frac{25}{x-5}$. Hence, $x - 5$ divides 25 and the possible values of $x-5$ are ± 1, ± 5, and ± 25. Since $x \neq 0$, then $x - 5 \neq -5$. The other values of $x - 5$ yield 6, 4, 10, 30, or -20 as possible values of x.

Each of these values for x generates an ordered pair of integers (x, y) which is a solution to the equation. These are $(6, 30)$, $(4, -20)$, $(10, 10)$, $(30, 6)$, and $(-20, 4)$.

In (b), we note that x cannot equal 0 or 1200, and y cannot equal 0 or 1200. Proceeding as in part (a) and solving the equation for y, we find that

$$y = \frac{1200x}{x-1200} = 1200 + \frac{1440000}{x-1200}.$$

Similarly, each divisor of 1440000 other than -1200 gives a possible value for $x - 1200$ and hence for x. Thus, the number of ordered pairs (x, y) satisfying $\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}$ is 1 less than the number of integer divisors of 1440000.

Now $1440000 = 2^8 \cdot 3^2 \cdot 5^4$, so 1440000 has $(8+1)(2+1)(4+1) = 135$ positive integer divisors. The number of integer pairs that are solutions is therefore $270 - 1 = 269$.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam (part (a) only); JACLYN CHANG, student, Western Canada High School, Calgary, AB (part (a) only); and NECULAI STANCIU, Saint Mucenic Sava Technological High School, Berca, Romania. There were 4 incorrect solutions submitted.