MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga) and Eric Robert (Leo Hayes High School, Fredericton).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **15 juin 2009**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M382. *Proposé par l'Équipe de Mayhem.*

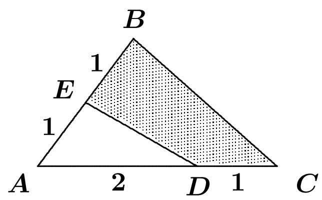
Déterminer toutes les paires (x, y) d'entiers tels que $4x^2 - y^2 = 480$.

M383. *Proposé par l'Équipe de Mayhem.*

Dans un rectangle $ABCD$, P est sur BC et Q est sur DC de sorte que $BP = 1$, $AP = PQ = 2$ et l'angle $APQ = 90°$. Déterminer la longueur de QD.

M384. *Proposé par Kunal Singh, étudiant, Kendriya Vidyalaya School, Shillong, Inde.*

Dans la figure ci-contre, le point E est sur AB et le point D est sur AC de sorte que $AE = EB = DC = 1$ et $AD = 2$. Déterminer le rapport de l'aire du quadrilatère $BCDE$ à celle du triangle ABC.



M385. *Proposé par Mihály Bencze, Brasov, Roumanie.*

En base 10, l'entier $N = 1 \cdots 114 \cdots 44$ commence avec 2009 chiffres 1 consécutifs suivis de 4018 chiffres 4 consécutifs. Montrer que N n'est pas un carré parfait.

M386. *Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucenic Sava, Berca, Roumanie.*

Déterminer tous les nombres réels x pour lesquels

$$\sqrt{2+4x-2x^2} + \sqrt{6+6x-3x^2} = x^2-2x+6\,.$$

M387. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

On peut mesurer la température en degrés Fahrenheit (F) ou en degrés Celsius (C). Les deux échelles sont reliées par la formule $F = 1.8C + 32$. Lorsqu'on convertit en Fahrenheit une température exprimée en Celsius par un nombre de deux chiffres, on constate parfois que, une fois les Fahrenheit arrondis à l'entier le plus proche, les chiffres des dizaines et des unités ont été permutés. Trouver toutes les valeurs entières de deux chiffres en C pour lesquelles ceci arrive.

M382. *Proposed by the Mayhem Staff.*

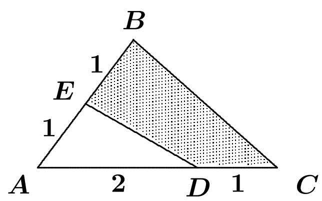
Determine all pairs (x,y) of integers for which $4x^2 - y^2 = 480$.

M383. *Proposed by the Mayhem Staff.*

In rectangle $ABCD$, P is on side BC and Q is on side DC so that $BP = 1$, $AP = PQ = 2$ and $\angle APQ = 90^\circ$. Determine the length of QD.

M384. *Proposed by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.*

In the diagram at right, the point E is on AB and the point D is on AC such that $AE = EB = DC = 1$ and $AD = 2$. Determine the ratio of the area of quadrilateral $BCDE$ to the area of triangle ABC.



M385. *Proposed by Mihály Bencze, Brasov, Romania.*

The base 10 integer $N = 1\cdots 114\cdots 44$ starts off with 2009 consecutive digits 1 followed by 4018 consecutive digits 4. Prove that N is not a perfect square.

M386. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

Determine all real numbers x for which

$$\sqrt{2+4x-2x^2} \;+\; \sqrt{6+6x-3x^2} \;=\; x^2-2x+6\,.$$

M387. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Temperature can be measured in degrees Fahrenheit (F) or in degrees Celsius (C). The two scales are related by the formula $F = 1.8C + 32$. When a two-digit integer degree temperature in Celsius is converted to Fahrenheit and rounded to the nearest integer degree, it turns out the ones and tens digits of the original Celsius temperature C sometimes switch places to give the rounded Fahrenheit equivalent. Find all two-digit integer values of C for which this occurs.

Mayhem Solutions

M344. *Proposed by the Mayhem Staff.*

Consider the square array

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

formed by listing the numbers 1 to 9 in order in consecutive rows. The sum of the integers on each diagonal is 15. If a similar array is constructed using the integers 1 to $10\,000$, what is the sum of the numbers on each diagonal?

Solution by José Hernández Santiago, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico, modified by the editor.

Let S_1 represent the sum of the integers on the diagonal that runs from top left to bottom right. Let S_2 represent the sum of the integers on the diagonal that runs from top right to bottom left.

Our array is 100 by 100. When we move one column to the right in the same row, the number increases by 1; when we move one column to the left in the same row, the number decreases by 1. When we move one row down in the same column, the number increases by 100.

On the top left to bottom right diagonal, each number is one column to the right and one row down, so is $1 + 100 = 101$ greater than the previous number on the diagonal. Thus,

$$\begin{aligned} S_1 &= 1 + 102 + 203 + \cdots + 9899 + 10000 \\ &= (100 \cdot 0 + 1) + (100 \cdot 1 + 2) + \cdots + (100 \cdot 98 + 99) \\ &\quad + (100 \cdot 99 + 100) \\ &= 100(0 + 1 + \cdots + 98 + 99) \ + \ (1 + 2 + \cdots + 99 + 100) \\ &= 100\left(\tfrac{1}{2}(99)(100)\right) \ + \ \tfrac{1}{2}(100)(101) \\ &= 100(99)(50) + 50(101) \ = \ 495000 + 5050 \ = \ 500050\,. \end{aligned}$$

On the top right to bottom left diagonal, each number is one column to the left and one row down, so is $-1 + 100 = 99$ greater than the previous number on the diagonal. Thus,

$$\begin{aligned} S_2 &= 100 + 199 + 298 + \cdots + 9802 + 9901 \\ &= (100(0) + 100) + (100(1) + 99) + \cdots + (100(98) + 2) \\ &\quad + (100(99) + 1) \\ &= 100(0 + 1 + \cdots + 98 + 99) \ + \ (100 + 99 + \cdots + 2 + 1)\,. \end{aligned}$$

Therefore, $S_2 = S_1 = 50050$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M345. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The area of isosceles $\triangle ABC$ is $q\sqrt{15}$. Given that $AB = 2BC$, express the perimeter of $\triangle ABC$ in terms of q.

Solution by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.

Let $BC = x$ and $AB = 2x$. Since $\triangle ABC$ is isosceles, then $AC = BC$ or $AC = AB$. Since $\triangle ABC$ is presumably not a degenerate triangle, we take $AC = AB = 2x$ (otherwise $AC = BC = x$ and $AB = 2x$, which would mean that the triangle was a straight line segment of length $2x$).

Let D be the midpoint of BC. Since $\triangle ABC$ is isosceles, then AD is perpendicular to BC. The length of this altitude is thus

$$\begin{aligned} AD &= \sqrt{AB^2 - \left(\tfrac{1}{2}BC\right)^2} \ = \ \sqrt{(2x)^2 - \left(\tfrac{1}{2}x\right)^2} \\ &= \sqrt{4x^2 - \tfrac{1}{4}x^2} \ = \ \sqrt{\tfrac{15}{4}x^2} \ = \ \frac{\sqrt{15}}{2}x\,. \end{aligned}$$

The area of $\triangle ABC$ is $q\sqrt{15}$ and also equal to $\frac{1}{2}(BC)(AD) = \frac{1}{2}x\left(\frac{\sqrt{15}}{2}x\right)$, so that $q\sqrt{15} = \frac{\sqrt{15}}{4}x^2$, or $x^2 = 4q$ and hence $x = 2\sqrt{q}$.

Thus, $AB + BC + CA = 2x + x + 2x = 5x = 10\sqrt{q}$ is the perimeter of the triangle.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were 2 incorrect solutions submitted.

M346. *Proposed by the Mayhem Staff.*

Without using a calculator, find the number of digits in the integer 2^{80}.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.

We determine the positive integer n for which $10^n \leq 2^{80} < 10^{n+1}$. This will imply that the integer 2^{80} has $n+1$ digits.

Since $1000 = 10^3 < 2^{10} = 1024$, raising both sides to the exponent 8 we obtain $10^{24} < 2^{80}$.

Also, $65536 = 2^{16} < 10^5 = 100000$, hence $2^{80} < 10^{25}$ upon raising both sides to the exponent 5.

Thus, $10^{24} < 2^{80} < 10^{25}$, whence 2^{80} has 25 digits.

Also solved by RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were 5 incomplete solutions submitted.

Singh submitted a solution that included more of a thought process that led to the same outcome as that of Ajanovic.

M347. *Proposed by the Mayhem Staff.*

Four positive integers a, b, c, and d are such that

$$\begin{aligned}(a+b+c)d &= 420\,,\\ (a+c+d)b &= 403\,,\\ (a+b+d)c &= 363\,,\\ (b+c+d)a &= 228\,.\end{aligned}$$

Find the four integers.

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

First, we write out the prime factorizations of the right sides of the equations:

$$\begin{aligned}420 &= 2^2\cdot 3\cdot 5\cdot 7\,, & 403 &= 13\cdot 31\,,\\ 363 &= 3\cdot 11^2\,, & 228 &= 2^2\cdot 3\cdot 19\,.\end{aligned}$$

Next, we subtract the second equation from the first equation to obtain $(a + c)d + bd - (a + c)b - bd = 420 - 403$, or $(a + c)(d - b) = 17$. Since a, b, c, and d are positive integers, then $a + c > 1$. Since $(a + c)$ is a divisor of 17, which is a prime number, then $a + c = 17$ and so $d - b = 1$.

From the third equation, c is odd since it is a divisor of 363. Since $a + c = 17$, then $a = 17 - c$ so a is even. Also, $1 \leq c \leq 15$ and $2 \leq a \leq 16$.

Since a is an even divisor of $228 = 2^2 \cdot 3 \cdot 19$ between 2 and 16 inclusive, then a could be 2, 4, 6, or 12. Since c is an odd divisor of $363 = 3 \cdot 11^2$ between 1 and 15 inclusive, then c could be 1, 3, or 11. Since $a + c = 17$, then a must equal 6 and c must equal 11.

So we know that $a = 6$, $c = 11$, and $d = b + 1$. Substituting these into the second equation, we obtain

$$\begin{aligned} (6 + 11 + b + 1)b &= 403\,; \\ (b + 18)b &= 403\,; \\ b^2 + 18b - 403 &= 0\,; \\ (b - 13)(b + 31) &= 0\,. \end{aligned}$$

Since b is a positive integer, we have $b = 13$, whence $d = 14$.

Therefore, $(a, b, c, d) = (6, 13, 11, 14)$, which we can check satisfies each of the four equations.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were 2 incomplete solutions and 1 incorrect solution submitted.

M348. *Proposed by the Mayhem Staff.*

The perimeter of a sector of a circle is 12 (the perimeter includes the two radii and the arc). Determine the radius of the circle that maximizes the area of the sector.

Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia, modified by the editor.

Let r be the radius of the circle, let θ be the central angle of the sector measured in radians, and let A be the area of the sector.

The length of the arc of the sector is $\frac{\theta}{2\pi}(2\pi r) = \theta r$. Since the perimeter of the sector is 12, then the arc length of the sector also equals $12 - 2r$ (by subtracting the total length of the two radii). Therefore, $\theta r = 12 - 2r$.

Also, the area of the sector is $A = \frac{\theta}{2\pi}\left(\pi r^2\right) = \frac{1}{2}\theta r^2$. Thus, we have $A = \frac{1}{2}(\theta r)r = \frac{1}{2}(12 - 2r)r = 6r - r^2$.

The vertex of the parabola defined by this expression corresponds to the maximum value for A. This occurs when $r = -\frac{6}{2(-1)} = 3$. Therefore, the radius of the circle that maximizes the area of the sector is $r = 3$ (which gives a maximum area of $A = 9$).

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M349. *Proposed by the Mayhem Staff.*

(a) Find all ordered pairs of integers (x, y) with $\frac{1}{x} + \frac{1}{y} = \frac{1}{5}$.

(b) How many ordered pairs of integers (x, y) are there with

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}?$$

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

First, in (a) note that x cannot equal 0 or 5, and y cannot equal 0 or 5, since neither the denominators nor the individual fractions can equal 0. Solving the equation for y, we find that $\frac{1}{y} = \frac{1}{5} - \frac{1}{x} = \frac{x-5}{5x}$ or

$$y = \frac{5x}{x-5} = \frac{(5x-25)+25}{x-5} = 5 + \frac{25}{x-5}.$$

Since y is an integer, then so is $\frac{25}{x-5}$. Hence, $x - 5$ divides 25 and the possible values of $x-5$ are ± 1, ± 5, and ± 25. Since $x \neq 0$, then $x - 5 \neq -5$. The other values of $x - 5$ yield 6, 4, 10, 30, or -20 as possible values of x.

Each of these values for x generates an ordered pair of integers (x, y) which is a solution to the equation. These are $(6, 30)$, $(4, -20)$, $(10, 10)$, $(30, 6)$, and $(-20, 4)$.

In (b), we note that x cannot equal 0 or 1200, and y cannot equal 0 or 1200. Proceeding as in part (a) and solving the equation for y, we find that

$$y = \frac{1200x}{x-1200} = 1200 + \frac{1440000}{x-1200}.$$

Similarly, each divisor of 1440000 other than -1200 gives a possible value for $x - 1200$ and hence for x. Thus, the number of ordered pairs (x, y) satisfying $\frac{1}{x} + \frac{1}{y} = \frac{1}{1200}$ is 1 less than the number of integer divisors of 1440000.

Now $1440000 = 2^8 \cdot 3^2 \cdot 5^4$, so 1440000 has $(8+1)(2+1)(4+1) = 135$ positive integer divisors. The number of integer pairs that are solutions is therefore $270 - 1 = 269$.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam (part (a) only); JACLYN CHANG, student, Western Canada High School, Calgary, AB (part (a) only); and NECULAI STANCIU, Saint Mucenic Sava Technological High School, Berca, Romania. There were 4 incorrect solutions submitted.

Problem of the Month

Ian VanderBurgh

Et tu, Brute force?

Problem (2008 Cayley Contest) The average value of

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-g)^2$$

over all possible arrangements (a, b, c, d, e, f, g) of the seven numbers 1, 2, 3, 11, 12, 13, 14 is

(A) 398 (B) 400 (C) 396 (D) 392 (E) 394

We learn how to calculate averages early on in our mathematics careers – add up all of the values and divide by the number of values. This isn't so hard when you're trying to calculate the average of your six marks at school, but can be a real pain if there are significantly more values to consider.

In this problem there are $7! = 7(6)(5)(4)(3)(2)(1) = 5040$ possible arrangements of the seven numbers 1, 2, 3, 11, 12, 13, 14. We could try to calculate the 5040 required values of

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-g)^2\,,$$

add them up, and divide by 5040. I think that you will agree that, in principle, we could do this calculation (the hard way!) by hand. Yes, it would take a very long time. Yes, we would be liable to make a whole bunch of arithmetic mistakes. Yes, it would be extremely annoying. But, yes, we could do it, as the underlying mathematics is not that hard. There must be a better way!

One approach would be to try to add the 5040 values without actually having to calculate the 5040 values. If we did this, we could then divide the total by 5040 and get our answer.

Put another way, we can think of the brute force approach of computing the value of

$$(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-g)^2 \qquad (*)$$

for each of the 5040 arrangements and then adding these values as "adding across then adding down". (In other words, compute each value and then add up the column of values.) But addition is commutative, that is, we don't have to add the values in the given order to get the correct total, so we could even break up the values into components and add these in separately. Let's give this a try.

Solution To determine the average value of the expression in $(*)$ we determine the sum of the values of this expression over all possible arrangements, and then divide by the number of arrangements. We determine the sum of all of the values of $(*)$ by examining the contribution of each possible term.

Let x and y be 2 of the 7 given numbers. In how many of these arrangements are x and y adjacent? Treat x and y as a single unit (xy) with 5 other numbers to be placed on either side of, but not between, xy. This gives 6 things (xy as a single unit and the 5 remaining numbers) to arrange, which can be done in $6(5)(4)(3)(2)(1)$ or $6!$ ways. But y could be followed by x, so there are $2(6!)$ arrangements with x and y adjacent, since there are the same number of arrangements with x followed by y as there are with y followed by x.

Since we want the sum of all of the possible values of $(*)$, we can calculate the total contribution of each possible term $(x-y)^2$ and add up these contributions. When we add up the values of $(*)$ over all possible arrangements, the term $(x-y)^2$ (which is equal to $(y-x)^2$) will occur $2(6!)$ times. This is true for any pair x and y. Thus, the sum of all of the possible values of $(*)$ must be equal to $2(6!)$ times the sum of all possible values of $(x-y)^2$.

The sum of all possible values of $(x-y)^2$ is

$$\begin{array}{cccccccccccc} 1^2 & + & 2^2 & + & 10^2 & + & 11^2 & + & 12^2 & + & 13^2 & \\ & + & 1^2 & + & 9^2 & + & 10^2 & + & 11^2 & + & 12^2 & \\ & & & + & 8^2 & + & 9^2 & + & 10^2 & + & 11^2 & \\ & & & & & + & 1^2 & + & 2^2 & + & 3^2 & \\ & & & & & & & + & 1^2 & + & 2^2 & \\ & & & & & & & & & + & 1^2 & = \; 1372\,. \end{array}$$

Here, we have paired 1 with each of the 6 larger numbers, then 2 with each of the 5 larger numbers, and so on. We only need to pair each number with all of the larger numbers because we have accounted for the reversed pairs in our method above.

Therefore, $2(6!)$ times the sum of $(x-y)^2$ over all choices of x and y with $x<y$, divided by $7!$ is the average value. This average value is

$$\frac{2(6!)(1372)}{7!} = \frac{2(1372)}{7} = 392\,.$$

This is one of the powerful things about mathematics – being able to turn a problem that looks as if it is difficult to solve in a short period of time into one that has a reasonably quick solution. We'll see another such problem in a couple of months.

There is an interesting footnote to this problem. When creating a problem, but especially a multiple choice problem, it's not good to be able to get the right answer for the wrong reason. As the CEMC was developing this problem, the fact it was multiple choice meant fiddling the actual numbers to avoid this issue. Try redoing this problem with 1, 2, 3, 4, 5, 6, 7, 8 and a suitably modified expression. You should get the answer 84. This is also the answer you'd get by assuming that the average value of any one of the squared terms is the average of $(5-4)^2$, $(6-3)^2$, $(7-2)^2$, and $(8-1)^2$. This would be a curious wrong way to get the right answer.