SKOLIAD No. 115

Lily Yen and Mogens Hansen

Please send solutions to problems in this Skoliad by **September 1, 2009**. Solutions should be sent to Lily Yen and Mogens Hansen at the address inside the back cover. A copy of ***MATHEMATICAL MAYHEM*** will be presented to the pre-university reader who sends in the best solution(s) before the deadline. The decision of the editors is final.

The Skoliad is in transition and, unfortunately, some submitted solutions have been lost. Our apologies for this inconvenience. Please resubmit any solutions to contests appearing in Skoliad in or after the March 2008 issue of ***CRUX***.

Our featured contest for this month is the British Columbia Secondary School Mathematics Contest 2007, Final Round, Part B. Our thanks go to Clint Lee, Okanagan College, Vernon, BC, for permission to use it.

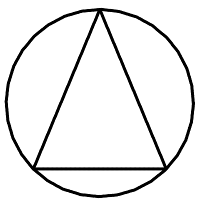
La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB, d'avoir traduit ce concours.

Concours mathématique des écoles secondaires de la Colombie-Britannique 2007
Ronde finale, partie B

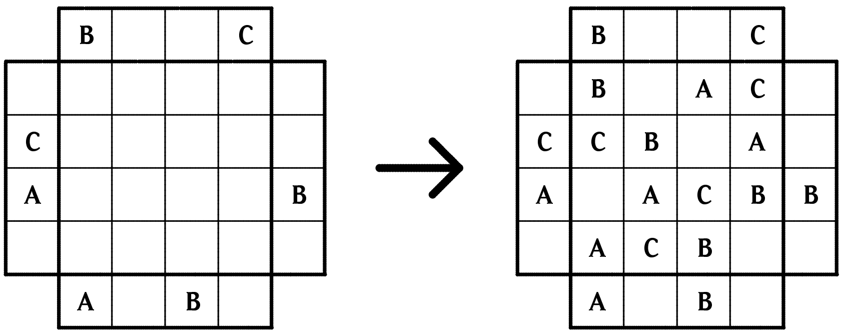
1. Jeanne a une collection de pièces de monnaie de 5¢, de 10¢ et de 25¢, dont la valeur totale est de $2,00$\$. Si les pièces de 5¢ étaient de 10¢ et les pièces de 10¢ étaient de 5¢, alors la valeur de la collection serait de $1,70$\$. Déterminer toutes les possibilités de nombres de pièces de 5¢, de 10¢ et de 25¢ dans la collection de Jeanne.

2. Un cube $3 \times 3 \times 3$ est formé en empilant des cubes $1 \times 1 \times 1$. Déterminer le nombre total de cubes, dont les côtés sont de taille entière, contenus dans le cube $3 \times 3 \times 3$.

3. Les longueurs des côtés d'un triangle sont 13, 13 et 10. Le cercle circonscrit de ce triangle est le cercle passant par les trois sommets du triangle et ayant ici son centre à l'intérieur du triangle (voir le diagramme à droite). Déterminer le rayon du cercle circonscrit.



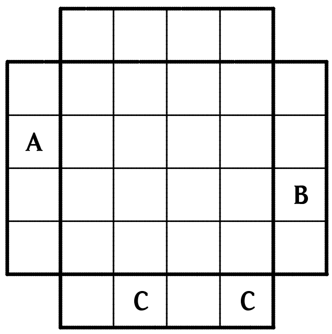
4. Le jeu de Latin se joue sur un tableau formé d'une grille quatre par quatre, avec une rangée additionnelle en haut comme en bas, puis une colonne additionnelle à gauche comme à droite. Les lettres A, B et C sont placées dans les cases de la grille quatre par quatre, de façon à ce que chaque lettre se retrouve exactement une fois dans chaque rangée et exactement une fois dans chaque colonne. En conséquence chaque rangée, de même que chaque colonne, aura exactement une case vide. Des lettres sont placées, comme indices de solution, à certains endroits dans les rangées et colonnes additionnelles ; elles indiquent la lettre la plus près se retrouvant dans la rangée ou colonne de la grille contenant l'indice. Le diagramme suivant donne une position de départ du jeu de Latin et la solution qui en résulte.



Debut **Solution**

Le diagramme d'un autre jeu de Latin est fourni à droite.

Compléter ce tableau, en fournissant une solution complète. Donner une justification des étapes permettant d'obtenir votre solution.



5. Déterminer toutes les solutions x et y, entières et positives, à l'équation

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}\,.$$

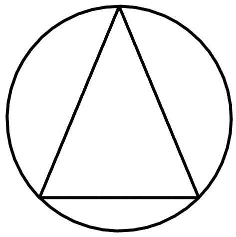
British Columbia Secondary School Mathematics Contest 2007

Final Round, Part B

1. Joan has a collection of nickels, dimes, and quarters worth $2.00. If the nickels were dimes and the dimes were nickels, the value of the coins would be $1.70. Determine all of the possibilities for the number of nickels, dimes, and quarters that Joan could have.

2. A $3 \times 3 \times 3$ cube is formed by stacking $1 \times 1 \times 1$ cubes. Determine the total number of cubes with sides of integral length that are contained in the $3 \times 3 \times 3$ cube.

3. The lengths of the sides of a triangle are 13, 13, and 10. The circumscribed circle of a triangle is the circle that goes through each of the three vertices of the triangle and here has its centre inside the triangle (see the diagram at right). Find the radius of the circumscribed circle.



4. The game of End View consists of a tableau with a four by four grid, one additional row at the top and at the bottom, and one additional column on the right and on the left. The letters A, B, and C are placed in the four by four grid in such a way that every letter appears exactly once in each row and each column. This means that there will be exactly one empty square in each row and each column. Letters are placed in the additional rows and columns as hints, at the end of some rows and columns of the four by four grid, to indicate the nearest letter that can be found by reading that row or column of the grid. The diagram below shows the starting tableau and the resulting solution tableau for a game of End View.

	B			C	
C					
A					B
	A		B		

Start

→

	B			C	
	B		A	C	
C	C	B		A	
A		A	C	B	B
	A	C	B		
	A		B		

Solution

The diagram for another game of End View is shown at right.

Fill in this tableau with the complete solution. Give a justification of the steps that you used to find the solution.

A					
					B
		C		C	

5. Determine all of the positive integer solutions, x and y, to the equation

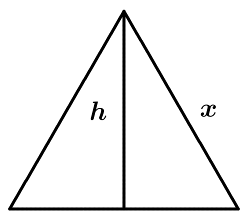
$$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$$

Now we will give solutions to the British Columbia Secondary School Mathematics Contest 2006, Junior Final Round, Part B [2008 : 129-130]. This time all the solutions are based on in-class work by students of the editors. Unfortunately, the solutions below were already typeset when we received a batch of (recovered) solutions from our readers. Thus, we could not feature any reader's solution this time.

1. Equilateral triangles I, II, III, and IV are such that the altitude of triangle I is the side of triangle II, the altitude of triangle II is the side of triangle III, and the altitude of triangle III is the side of triangle IV. If the area of triangle I is 2, find the area of triangle IV.

Solution.

If the side length of triangle I is x_1, then by the Pythagorean Theorem the height, h_1, of triangle I is $\frac{\sqrt{3}}{2}x_1$. But then the side length, x_2, of triangle II is $\frac{\sqrt{3}}{2}x_1$ and the height, h_2, of triangle II is $\frac{\sqrt{3}}{2}h_1$. Therefore the area, A_2, of triangle II is $\frac{x_2h_2}{2} = \left(\frac{\sqrt{3}}{2}\right)^2 \frac{x_1h_1}{2} = \frac{3}{4}A_1$, where A_1 is the area of triangle I. Similarly, $A_3 = \frac{3}{4}A_2$ and $A_4 = \frac{3}{4}A_3$, where A_3 and A_4 are the areas of triangle III and triangle IV, respectively. Since $A_1 = 2$, we finally obtain $A_4 = \left(\frac{3}{4}\right)^3 A_1 = \frac{27}{32}$.



Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; KARTHIK NATARAJAN; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

2. A square has an area of 3 square units, and a cube has a volume of 5 cubic units. Which is larger, the edge length of the square or the edge length of the cube? Justify your answer using the exact values of the two quantities.

Solution.

Let s be the side length of the square, and let c be the side length of the cube. Then $s^2 = 3$ and $c^3 = 5$, so $s^6 = 3^3 = 27$ and $c^6 = 5^2 = 25$, whence $s > c$.

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; KARTHIK NATARAJAN; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

3. A certain positive integer has "6" as its last (rightmost) digit. This number is transformed into a new number by moving the "6" to the beginning of the number (leftmost position). For example, the number 1236 would be transformed to 6123, while 51476 becomes 65147. What is the smallest such positive integer for which this transformation increases the value of the number by a factor of 4?

Solution.

Say the integer is $\dots A6$, that is the rightmost two digits are A and 6. Then $4(\dots A6) = 6\dots A$. But since $4 \cdot 6 = 24$, the unit digit of $4(\dots A6)$ must be 4. Thus $A = 4$ and the integer must be $\dots B46$. Since $4 \cdot 46 = 184$ and $4(\dots B46) = 6\dots B4$, you have that $B = 8$ and the integer must be $\dots C846$. Now, $4 \cdot 846 = 3384$ and $4(\dots C846) = 6\dots C84$, so $C = 3$ and the integer is $\dots D3846$. Again, $4 \cdot 3846 = 15384$ and so we have $4(\dots D3846) = 6\dots D384$, so $D = 5$ and the integer is $\dots E53846$. Mustering all our patience, $4 \cdot 53846 = 215384$ and $4(\dots E53846) = 6\dots E5384$, so $E = 1$ and the integer is $\dots F153846$. Since $4 \cdot 153846 = 615384$—that's it, the rightmost six digits of the required integer must be 153846. Hence, the smallest such integer is 153846.

Also solved by KARTHIK NATARAJAN.

4. The members of a committee sit at a circular table so that each committee member has two neighbours. Each member of the committee has a certain number of dollars in his or her wallet. The chairperson of the committee has one more dollar than the vice chairperson, who sits on his right and has one more dollar than the member on her right, who has one more dollar than the person on his right, and so on, until the member on the chair's left is reached. The chairperson now gives one dollar to the vice chair, who gives two dollars to the member on her right, who gives three dollars to the member on his right, and so on, until the member on the chair's left is reached. There are then two neighbours, one of whom has four times as much as the other.

(a) What is the smallest possible number of members of the committee? In this case, how much did the poorest member of the committee have at first?

(b) If there are at least 12 members of the committee, what is the smallest possible number of members of the committee? In this case, how much did the poorest member of the committee have at first?

Solution.

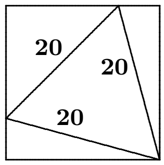
Say the committee has n members and the poorest member has x dollars. Then the chair has $x+n-1$ dollars, the vice chair has $x+n-2$ dollars, and so on. Each person, except the poorest, receives one fewer dollar than he or she gives away, so each person, except the poorest, loses one dollar. Consequently, the poorest person gains $n-1$ dollars. The chair therefore now has $x+n-2$ dollars, the vice chair has $x+n-3$ dollars, and so on until the person who used to be the second poorest now has x dollars, while the person who used to be poorest finds him- or herself with $x+n-1$ dollars. Thus the dollar amounts have simply shifted by one person around the table. The difference in dollar amounts between neighbours is therefore always one dollar, except between the poorest and the wealthiest person. A difference of one dollar cannot quadruple an integer amount of dollars, so $x+n-1=4x$. But then $n-1=3x$, so $n-1$ is a multiple of 3.

For part (a), you can use $n=4$ and $x=1$, so the smallest possible committee has four members and the poorest member has one dollar.

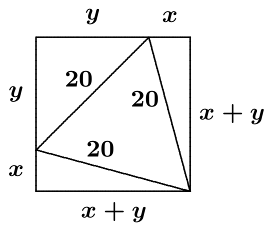
For part (b), you can use $n=13$ and $x=4$, so the smallest committee with 12 or more members has 13 members.

Also solved by KARTHIK NATARAJAN.

5. An equilateral triangle, 20 centimetres on a side, is inscribed in a square, as shown in the diagram. Find the length of the side of the square.



Solution.



The two smaller triangles in the figure both have a right angle, a hypotenuse of length 20, and a side shared with the square. Therefore the third sides of those triangles are equal, and you may label the pieces as shown.

Using the Pythagorean Theorem twice, $2y^2 = 20^2$ and $(x+y)^2 + x^2 = 20^2$, so $y = \pm 10\sqrt{2}$ and $2x^2 + 2xy + y^2 = 400$. Since y is a length, $y = 10\sqrt{2}$ and $x^2 + xy - 100 = 0$. The quadratic formula now yields

$$x = \frac{-y \pm \sqrt{y^2+400}}{2} = \frac{-10\sqrt{2} \pm 10\sqrt{6}}{2} = -5\sqrt{2} \pm 5\sqrt{6}\,.$$

Since x is also a length, $x = 5\sqrt{6} - 5\sqrt{2}$, and therefore the side length of the square is $x + y = 5\sqrt{6} + 5\sqrt{2}$.

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; KARTHIK NATARAJAN; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

We have also recovered a correct solution to problem #4 of the Mathematics Association of Quebec Contest 2006 at [2008 : 66, 68] by Luyun Zhong-Qiao, Columbia International College, Hamilton, ON.