SOLUTIONS

Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.

3313. [2008 : 104, 106] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let $x_1, x_2, \dots, x_n$ be real numbers such that $x_k > 1$ for $1 \le k \le n$. If we set $x_{n+1} = x_1$, prove that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\log_{x_k} x_{k+1} + \log_{x_{k+1}} x_k\right) \;\le\; \left(\prod_{k=1}^{n}\left(1+\log^n_{x_k} x_{k+1}\right)\right)^{\frac{1}{n}}.$$

Similar solutions by George Apostolopoulos, Messolonghi, Greece; Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA; and the proposers.

We prove the following more general result of which the given inequality is a special case: If $a_1, a_2, \dots, a_n$ are positive real numbers such that $\prod\limits_{k=1}^{n} a_k = 1$, then

$$\frac{1}{n}\left(\sum_{k=1}^{n} a_k + \sum_{k=1}^{n}\frac{1}{a_k}\right) \;\le\; \left(\prod_{k=1}^{n}\left(1+a_k^n\right)\right)^{1/n}. \tag{1}$$

Applying the AM–GM Inequality, we have for each fixed j, $1 \le j \le n$,

$$\left(\sum_{\substack{k=1\\k\neq j}}^{n}\frac{a_k^n}{1+a_k^n}\right) + \frac{1}{1+a_j^n} \;\ge\; \frac{n}{a_j\left(\prod\limits_{k=1}^{n}\left(1+a_k^n\right)\right)^{1/n}} \tag{2}$$

and

$$\left(\sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{1+a_k^n}\right) + \frac{a_j^n}{1+a_j^n} \;\ge\; \frac{na_j}{\left(\prod\limits_{k=1}^{n}\left(1+a_k\right)\right)^{1/n}}. \tag{3}$$

Adding the inequalities (2) and (3) we obtain

$$\left(a_j + \frac{1}{a_j}\right)\frac{n}{\left(\prod\limits_{k=1}^{n}\left(1+a_k\right)\right)^{1/n}} \;\le\; n\,,$$

or

$$a_j + \frac{1}{a_j} \;\le\; \left(\prod_{k=1}^{n} \left(1 + a_k^n\right) \right)^{1/n} . \tag{4}$$

Inequality (1) follows upon summing inequality (4) on $j = 1, 2, \ldots, n$.

If $a_k = \log_{x_k} x_{k+1}$, then $a_k > 0$ for all k and $\prod\limits_{k=1}^{n} a_k = \prod\limits_{k=1}^{n} \dfrac{\ln x_{k+1}}{\ln x_k} = 1$, so the proposed inequality follows from inequality (1).

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Both Oliver Geupel, Brühl, NRW, Germany and Ricardo pointed out that the proposed inequality is a special case of (1) *which was attributed to Gabriel Dospinescu and can be found in the book* Old and New Inequalities *by T. Andreescu, V. Cîrtoaje, G. Dospinescu, and M. Lascu (GIL Publishing House, Zalau, Romania, 2004).*

3314. [2008 : 102, 104] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\sum_{\text{cyclic}} \frac{a}{b} \;\ge\; \frac{3}{4} + \sum_{\text{cyclic}} \frac{(a+c)^2 + (a+b)c}{(b+c)(2a+b+c)} .$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

The solution makes use of the following two lemmas.

Lemma 1 For all positive real numbers x, y, and z,

$$\sum_{\text{cyclic}} \left(\frac{x}{y} - \frac{z+x}{z+y} \right) \;\ge\; 0 .$$

Proof: By a shift of the cycle (x, y, z) , we can force the number z to be the minimum value of x, y, and z. Then we have

$$\sum_{\text{cyclic}} \left(\frac{x}{y} - \frac{z+x}{z+y} \right) \;=\; \frac{(x-y)^2}{xy} + \frac{(x-z)(y-z)}{xz} - \frac{(x-y)^2}{(x+z)(y+z)} - \frac{(x-z)(y-z)}{(x+y)(z+x)} \;\ge\; 0 ,$$

as required. ∎

Lemma 2 For all positive real numbers a, b, and c,

$$\sum_{\text{cyclic}} \frac{(a+b)a}{(b+c)(2a+b+c)} \;\ge\; \frac{3}{4} . \tag{1}$$

Proof: By Lemma 1 and Nesbitt's Inequality, for all positive real numbers x, y, and z we have

$$\sum_{\text{cyclic}} \frac{x(x-y+z)}{y(x+z)} = \sum_{\text{cyclic}} \left(\frac{x}{y} - \frac{z+x}{z+y}\right) + \sum_{\text{cyclic}} \frac{y}{x+z} \geq \frac{3}{2}.$$

We let $x = a+b$, $y = b+c$, and $z = c+a$ to obtain the inequality (1). ∎

From the two lemmas, we conclude that

$$\begin{aligned}\sum_{\text{cyclic}} \frac{a}{b} &\geq \sum_{\text{cyclic}} \frac{c+a}{c+b} \\ &= \sum_{\text{cyclic}} \frac{(a+b)a}{(b+c)(2a+b+c)} + \sum_{\text{cyclic}} \frac{(a+b)^2+(a+b)c}{(b+c)(2a+b+c)} \\ &\geq \frac{3}{4} + \sum_{\text{cyclic}} \frac{(a+b)^2+(a+b)c}{(b+c)(2a+b+c)}.\end{aligned}$$

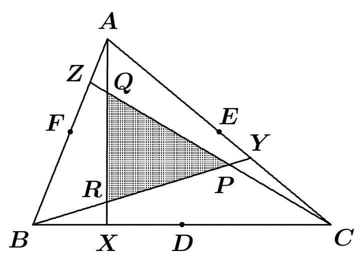
This is the desired inequality.

Also solved by the proposer.

Geupel indicated that Lemma 1 also appeared as an exercise in the Indian team selection test for the IMO 2002, see **Crux with Mayhem** *34 (2008) p. 151, Problem 20. Lemma 2 also appeared as Problem 28 on p. 203 in a compilation by Eckard Specht,* `http://www.imomath.com/othercomp/Journ/ineq.pdf` *with two references: T. Andreescu, V. Cîrtoaje, G. Dospinescu, and M. Lascu,* Old and New Inequalities, *and* Gazeta Matematica *[D. Olteanu]. He could not verify any of these sources. Indeed, Lemma 2 appears as problem 28, p. 11 of* Old and New Inequalities, *attributed there to D. Olteanu and as appearing in* Gazeta Matematica. *However, we could not verify the latter sources.*

3315. [2008 : 102, 105] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Let D, E, and F be the mid-points of sides BC, CA, and AB, respectively, in $\triangle ABC$. Let X, Y, and Z be points on the segments BD, CE, and AF, respectively. The lines AX, BY, and CZ bound a central triangle (shaded in the diagram). Let X', Y', and Z' be the reflections of X, Y, and Z in the points D, E, and F, respectively. The points X', Y', and Z' determine in a similar manner another central triangle $P'Q'R'$.



Prove that

$$\frac{2+\sqrt{3}}{4} \leq \frac{[PQR]}{[P'Q'R']} \leq 8 - 4\sqrt{3},$$

where $[STU]$ represents the area of $\triangle STU$.

Solution by Michel Bataille, Rouen, France.

Let $\mathcal{K} = [0,1]\times[0,1]\times[0,1]$ and let $x = \dfrac{BX}{XC}$, $y = \dfrac{CY}{YA}$, and $z = \dfrac{AZ}{ZB}$. Note that $(x,y,z) \in \mathcal{K}$ and that $x = \dfrac{CX'}{X'B}$, $y = \dfrac{AY'}{Y'C}$, and $z = \dfrac{BZ'}{Z'A}$.

Using Routh's Theorem (see problem 2752 [2002 : 329; 2003 : 331]), we see that $\rho(x,y,z) = \frac{[PQR]}{[P'Q'R']}$ is given by

$$\rho(x,y,z) \;=\; \frac{(1+x+zx)(1+y+xy)(1+z+yz)}{(1+z+zx)(1+x+xy)(1+y+yz)}\,.$$

The continuous function ρ attains its maximum M and its minimum m on the compact set $\mathcal{K}$; we will show that $m = \dfrac{2+\sqrt{3}}{4}$ and $M = 8 - 4\sqrt{3}$.

First, we restrict ρ to an edge of the cube $\mathcal{K}$, characterized by one variable equal to 0 and another equal to 1, say $x = 1$, $y = 0$. A quick study of $\phi(z) = \rho(1,0,z) = \dfrac{2+3z+z^2}{2(1+2z)}$ on $[0,1]$ shows that

$$1 \;\ge\; \phi(z) \;\ge\; \phi\left(\frac{\sqrt{3}-1}{2}\right) \;=\; \frac{2+\sqrt{3}}{4}\,.$$

Similarly, on the edge $x = 0$, $y = 1$, we have

$$1 \;\le\; \rho(0,1,z) \;=\; \frac{1}{\phi(z)} \;\le\; 8-4\sqrt{3} \;=\; \rho(0,1,\frac{\sqrt{3}-1}{2})\,.$$

It follows that $m \le \dfrac{2+\sqrt{3}}{4}$ and $M \ge 8 - 4\sqrt{3}$ and that it suffices to prove that M, m cannot be attained on either

(a) an edge of the cube $\mathcal{K}$ where two variables are equal,

(b) the interior of the cube $\mathcal{K}$, or

(c) the interior of a face of the cube $\mathcal{K}$.

A key remark is that $m < 1$ and $M > 1$, so that m, M cannot be attained at a triple containing two equal numbers, because ρ takes the value 1 at such a triple (it is easily checked, for example, that $\rho(x,x,z) = 1$). This remark settles case (a).

(b) Assume that ρ has an extremum at an interior point (x,y,z) of $\mathcal{K}$. Then by the previous case x, y, z are distinct and we have that $\dfrac{\partial\rho}{\partial x}$, $\dfrac{\partial\rho}{\partial y}$, and $\dfrac{\partial\rho}{\partial z}$ all vanish at (x,y,z). Using logarithmic derivatives, we may rewrite the latter condition as

$$\frac{p(z,z)}{p(x,z)p(z,x)} \;=\; \frac{p(y,y)}{p(x,y)p(y,x)}\,; \qquad \frac{p(z,z)}{p(y,z)p(z,y)} \;=\; \frac{p(x,x)}{p(x,y)p(y,x)}\,;$$

$$\frac{p(x,x)}{p(x,z)p(z,x)} \;=\; \frac{p(y,y)}{p(y,z)p(z,y)}\,;$$

where $p(u,v) = 1+u+uv$. Since $p(x,x)p(y,y) = p(x,y)p(y,x)+(x-y)^2$, the preceding equations are equivalent to

$$\left(1+x+x^2\right)(y-z)^2 \;=\; \left(1+y+y^2\right)(z-x)^2 \;=\; \left(1+z+z^2\right)(x-y)^2\,.$$

Solving the first equation for z leads to

$$(1+x+y)z+1-xy \;=\; \sqrt{\left(1+x+x^2\right)\left(1+y+y^2\right)}\,,$$

and similarly $(1+y+z)x+1-yz = \sqrt{\left(1+y+y^2\right)\left(1+z+z^2\right)}$. By subtracting these two expressions and dividing by $(z-x)$, we arrive at

$$\frac{\sqrt{1+y+y^2}}{\sqrt{1+x+x^2}+\sqrt{1+z+z^2}}(1+z+x) \;=\; -(1+2y)\,,$$

a contradiction since $1+z+x>0$ and $1+2y>0$ in the interior of $\mathcal{K}$.

(c) We can limit the study to the faces $x=0$ and $x=1$. Consider first $\rho(0,y,z) = \dfrac{(1+y)(1+z+yz)}{(1+z)(1+y+yz)}$. The necessary condition for an extremum now reads as

$$(1+y)\left(1+z+z^2\right) \;=\; (1+z)\left(1+y+y^2\right) \;=\; (1+z+yz)(1+y+yz)$$

and the first equation yields $(z-y)(z+y+yz)=0$, which is impossible when $y \neq z$. Similarly, the study of possible extrema for $\rho(1,y,z)$ leads to the condition $\dfrac{(2+y)(1+2y)}{1+y+y^2} = \dfrac{(2+z)(1+2z)}{1+z+z^2}$, again impossible for $y\neq z$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; and the proposer.

3316. [2008 : 103, 105] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\sum_{\text{cyclic}}\frac{a}{b}+\Big(\sum_{\text{cyclic}}a^2\Big)^{\frac{1}{2}}\Big(\sum_{\text{cyclic}}\frac{1}{a^2}\Big)^{\frac{1}{2}} \;\geq\; \frac{2}{3}\Big(\sum_{\text{cyclic}}a\Big)\Big(\sum_{\text{cyclic}}\frac{1}{a}\Big)\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Applying the Cauchy-Schwarz Inequality and the AM–GM Inequality, we obtain:

$$
\begin{aligned}
& \sum_{\text{cyclic}} \frac{a}{b} + \left(\sum_{\text{cyclic}} a^2\right)^{\frac{1}{2}} \left(\sum_{\text{cyclic}} \frac{1}{a^2}\right)^{\frac{1}{2}} \\
= & \left(\sum_{\text{cyclic}} \frac{a}{b}\right) + \left((a^2+b^2+c^2)\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)\right)^{\frac{1}{2}} \\
\geq & \left(\sum_{\text{cyclic}} \frac{a}{b}\right) + \left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right) \\
= & \frac{2}{3}\left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} \frac{1}{a}\right) + \frac{1}{3}\left(\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)-6\right) \\
\geq & \frac{2}{3}\left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} \frac{1}{a}\right) + \frac{1}{3}(2+2+2-6) \\
= & \frac{2}{3}\left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} \frac{1}{a}\right) .
\end{aligned}
$$

Equality holds if and only if $a = b = c$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3317. [2008 : 103, 105] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$
\begin{aligned}
& \left(\sum_{\text{cyclic}} \frac{a^3}{b^2-bc+c^2}\right)\left(\sum_{\text{cyclic}} \frac{b^2c^2}{a^3(b^2-bc+c^2)}\right) \\
& \quad \geq \left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} \frac{1}{a}\right) \geq 16\left(\sum_{\text{cyclic}} \frac{ab}{a+b+2c}\right)\left(\sum_{\text{cyclic}} \frac{c}{2ab+bc+ca}\right) .
\end{aligned}
$$

Similar solutions by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam and by the proposer.

It suffices for us to prove the two inequalities on the following page:

$$\sum_{\text{cyclic}} \frac{a^3}{b^2 - bc + c^2} \geq \sum_{\text{cyclic}} a \geq 4 \sum_{\text{cyclic}} \frac{ab}{a+b+2c}, \qquad (1)$$

$$\sum_{\text{cyclic}} \frac{b^2c^2}{a^3(b^2 - bc + c^2)} \geq \sum_{\text{cyclic}} \frac{1}{a} \geq 4 \sum_{\text{cyclic}} \frac{c}{2ab+bc+ca}. \qquad (2)$$

Hence,

$$\begin{aligned} 4 \sum_{\text{cyclic}} \frac{ab}{a+b+2c} &= 4 \sum_{\text{cyclic}} \frac{ab}{(c+a)+(c+b)} \\ &\leq \sum_{\text{cyclic}} \left(\frac{ab}{c+a} + \frac{ab}{c+b} \right) \\ &= \frac{c(a+b)}{a+b} + \frac{a(b+c)}{b+c} + \frac{b(c+a)}{c+a} = \sum_{\text{cyclic}} a. \end{aligned}$$

This establishes the last inequality of (1). By the Cauchy–Schwartz Inequality, we have

$$\begin{aligned} \sum_{\text{cyclic}} \frac{a^3}{(b^2 - bc + c^2)} &= \sum_{\text{cyclic}} \frac{a^4}{a(b^2 - bc + c^2)} \\ &\geq \left(\sum_{\text{cyclic}} a^2 \right)^2 \left(\sum_{\text{cyclic}} a(b^2 - bc + c^2) \right)^{-1}. \end{aligned}$$

However, the following inequalities are equivalent

$$\begin{aligned} \left(\sum_{\text{cyclic}} a^2 \right)^2 \left(\sum_{\text{cyclic}} a(b^2 - bc + c^2) \right)^{-1} &\geq \sum_{\text{cyclic}} a; \\ \left(\sum_{\text{cyclic}} a \right) \left(\sum_{\text{cyclic}} a(b^2 - bc + c^2) \right) &\leq \left(\sum_{\text{cyclic}} a^2 \right)^2; \\ \sum_{\text{cyclic}} a^4 + abc \sum_{\text{cyclic}} a &\geq \sum_{\text{cyclic}} ab(a^2 + b^2); \\ \sum_{\text{cyclic}} a^2(a-b)(a-c) &\geq 0. \end{aligned}$$

The last inequality is Schur's inequality, and this completes the proof of (1).

To prove (2) we again use the inequality $\frac{4}{x+y} \leq \left(\frac{1}{x} + \frac{1}{y}\right)$ for positive

x and y, as follows

$$\begin{aligned} 4\sum_{\text{cyclic}} \frac{c}{2ab+bc+ca} &= 4\sum_{\text{cyclic}} \frac{c}{(ab+bc)+(ab+ac)} \\ &\le \sum_{\text{cyclic}} \left(\frac{c}{b(a+c)} + \frac{c}{a(b+c)}\right) \\ &= \frac{(a+c)}{b(a+c)} + \frac{(b+c)}{a(b+c)} + \frac{(a+b)}{c(a+b)} = \sum_{\text{cyclic}} \frac{1}{a}\,. \end{aligned}$$

This proves the second inequality in (2).

By setting $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$, the first inequality in (2) becomes

$$\sum_{\text{cyclic}} \frac{x^3}{y^2 - yz + z^2} \ \ge\ \sum_{\text{cyclic}} x\,,$$

which follows from (1). This completes the proof of (2).

Equality holds if and only if $a = b = c$.

Also solved by ARKADY ALT, San Jose, CA, USA; and TITU ZVONARU, Comănești, Romania. There was one incorrect solution submitted.

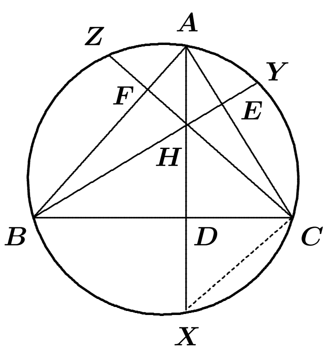
3318. [2008 : 103, 105] *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

The altitudes AD, BE, and CF of $\triangle ABC$ are produced to meet the circumcircle at X, Y, and Z, respectively. Prove that

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} \ =\ 4\,.$$

Similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; Ricardo Barroso Campos, University of Seville, Seville, Spain; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Apostolis K. Demis, Varvakeio High School, Athens, Greece; Andrea Munaro, student, University of Trento, Trento, Italy; and George Tsapakidis, Agrinio, Greece.

Let H be the orthocentre of $\triangle ABC$. The proof is trivial when the triangle ABC is right-angled. We now assume that $\triangle ABC$ is acute. The right triangles CDH and CDX are congruent, because



$$\begin{aligned} \angle DHC &= 90^\circ - \angle DCH \\ &= 90^\circ - \angle BCF \\ &= 90^\circ - (90^\circ - \angle B) \\ &= \angle B \ =\ \angle AXC \\ &= \angle DXC\,. \end{aligned}$$

Hence, $XD = HD$. Similarly, $YE = HE$ and $ZF = HF$. Let $[ABC]$ denote the area of triangle ABC. We have

$$\begin{aligned}\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} &= \frac{AD+XD}{AD} + \frac{BE+YE}{BE} + \frac{CF+ZF}{CF} \\ &= \frac{AD+HD}{AD} + \frac{BE+HE}{BE} + \frac{CF+HF}{CF} \\ &= 3 + \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} \\ &= 3 + \frac{[HBC]}{[ABC]} + \frac{[HAC]}{[ABC]} + \frac{[HAB]}{[ABC]} \\ &= 3 + \frac{[ABC]}{[ABC]} = 4\,.\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA (2 solutions); KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Bataille, Geupel, and Peiró gave solutions using signed distances, for which the featured solution is valid for all triangles, but the statement (as given) is not true for obtuse triangles. For example, if $AB = AC$, then AX is fixed (it is the diameter of the circumcircle), so the ratio $\frac{AX}{AD}$ can be larger than 4; in fact, it can be made arbitrarily large. The correct statement for an obtuse triangle, say, with $\angle A > 90^\circ$, B between E and Y, and C between F and Z would be

$$\frac{AX}{AD} - \frac{BY}{BE} - \frac{CZ}{CF} = 4\,.$$

3319. [2008 : 103, 106] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let m be a natural number, $m \geq 2$, and let r be any real number such that $r \geq 1/m$. If a and b are positive real numbers satisfying $ab = r^2$, prove that

$$\frac{1}{(1+a)^m} + \frac{1}{(1+b)^m} \geq \frac{2}{(1+r)^m}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

First we show that for all positive real numbers x,

$$\frac{(x+1)^2}{x^{2/(m+1)}} \geq \frac{m+1}{m-1} \cdot \frac{1-r^2}{r^{2/(m+1)}}\,. \qquad (1)$$

To prove (1), consider the function $h(x) = \frac{(x+1)^2}{x^{2/(m+1)}}$ for positive x. From the derivative

$$h'(x) = \frac{2(x+1)}{x^{(m+3)/(m+1)}}\left(x - \frac{x+1}{m+1}\right),$$

we see that $h'(x) \le 0$ for $x \in \left(0, \frac{1}{m}\right]$ while $h'(x) \ge 0$ for $x \in \left[\frac{1}{m}, \infty\right)$. Therefore, h takes its minimum value at $x = \frac{1}{m}$ and we have

$$h\left(\frac{1}{m}\right) = \frac{m+1}{m-1}\left(1 - \frac{1}{m^2}\right)m^{2/(m+1)} \ge \frac{m+1}{m-1}\cdot\frac{1-r^2}{r^{2/(m+1)}},$$

which completes the proof of (1).

Next we claim that

$$k(x) = r^2x^{m-1}(1+x)^{m+1} - \left(x+r^2\right)^{m+1}$$

is not positive if $0 < x \le r$ and is not negative if $x \ge r$. For this, consider the function $g(x) = r^{2/(m+1)}x^{(m-1)/(m+1)} - \frac{x+r^2}{x+1}$, for which we have

$$g'(x) = \frac{m-1}{m+1}\cdot\frac{r^{2/(m+1)}}{x^{2/(m+1)}} - \frac{1-r^2}{(x+1)^2}.$$

Observe that $g(r) = 0$ and, by (1), $g'(x) \ge 0$ for all $x > 0$, which implies our claim regarding $k(x)$.

Finally, we prove the required inequality by writing $x = a$, $b = \frac{r^2}{x}$, and by considering for $x > 0$ the function $f(x) = \frac{1}{(1+x)^m} + \frac{1}{(1+r^2/x)^m}$. We have

$$f'(x) = \frac{m}{\left((1+x)(x+r^2)\right)^{m+1}}\,k(x).$$

From what we know about $k(x)$ we have that $f'(x) \le 0$ for $0 < x \le r$, and $f'(x) \ge 0$ for $x \ge r$. Therefore, f takes its minimum value at $x = r$, which is $f(r) = \frac{2}{(1+r)^m}$, as desired.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete and three incorrect solutions submitted.

3320. [2008 : 103, 106] *Proposed by Michel Bataille, Rouen, France.*

Let $\triangle ABC$ be right-angled at A and let O be the midpoint of BC. Let M be a point in the plane of $\triangle ABC$, and let M', M'', N, N', and N'' denote the orthocentres of $\triangle MAB$, $\triangle MAC$, $\triangle AM'M''$, $\triangle NAB$, and $\triangle NAC$, respectively. If O is the midpoint of $M'M''$, show that O is also the midpoint of $N'N''$.

Solution by the proposer.

Let us assume that M lies on neither AB nor AC so that M' and M'' will be well defined. We first find the locus of points M such that O is the midpoint of $M'M''$. To this end we choose a system of axes with origin at O, such that the vertices have coordinates $A(-b,-c)$, $B(b,-c)$, and $C(-b,c)$. Let M have coordinates $M(x,y)$. Since $MM' \perp AB$ and $MM'' \perp AC$, we have coordinates $M'(x,\beta)$ and $M''(\alpha,y)$ for some pair of real numbers α and β. For BM' and CM'' perpendicular to AM we obtain

$$(x+b)(x-b)+(y+c)(\beta+c) \;=\; 0 \;=\; (x+b)(\alpha+b)+(y+c)(y-c)\,. \quad (1)$$

If O is the midpoint of $M'M''$, then $\alpha = -x$, $\beta = -y$, and (1) becomes

$$x^2 - y^2 \;=\; b^2 - c^2\,. \quad (2)$$

Conversely, if (2) holds then by (1) we have $(\alpha+b)(x+b) = c^2-y^2 = b^2-x^2$, which (since we assume $x \neq -b$) implies $\alpha = -x$. Similarly, $\beta = -y$ and O is the midpoint of MM'. Note that (2) is the equation of a rectangular hyperbola $\mathcal{H}$ which contains A, B, and C. (It degenerates into the pair of lines OA and BC if $\triangle ABC$ is isosceles, in which case $b = \pm c$.)

We now appeal to the standard theorem that if a triangle is inscribed in a rectangular hyperbola, its orthocentre is also on the hyperbola: If O is the midpoint of $M'M''$, then M is on $\mathcal{H}$ (as we have just seen), and since A and B are also on $\mathcal{H}$, we conclude that M' is on $\mathcal{H}$ as well. Similarly M'' is on $\mathcal{H}$, and so $AM'M''$ forms a triangle that is inscribed in $\mathcal{H}$. As a consequence, N is likewise on $\mathcal{H}$, whence O is the midpoint of $N'N''$. This concludes the argument for all positions of M where the M's and N's are all well defined; the remaining positions can be handled easily by noting that as M moves continuously along $\mathcal{H}$, so do N, N', and N''.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3321. [2008 : 103, 108] *Proposed by Michel Bataille, Rouen, France.*

Let the incircle of $\triangle ABC$ have centre I and meet the sides AC and AB at E and F, respectively. For a point M on the line segment EF, show that $\triangle MAB$ and $\triangle MCA$ have the same area if and only if $MI \perp BC$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let P and Q be the projections of M on the sides AC and AB, respectively. Note that because M is on the segment EF, it lies inside $\triangle ABC$; moreover, because $AF = AE$ the right triangles MQF and MPE are similar, so that

$$\frac{MQ}{MP} \;=\; \frac{MF}{ME}\,. \quad (1)$$

Triangles MAB and MCA have equal areas if and only if $c \cdot MQ = b \cdot MP$, or $\frac{b}{c} = \frac{MQ}{MP}$. Consequently, from (1) we see that it is sufficient to prove that MI is perpendicular to BC if and only if

$$\frac{b}{c} = \frac{MF}{ME}\,. \tag{2}$$

Let D be the projection of I on BC. Suppose first that M lies on ID. Since $BDIF$ is cyclic, $\angle MIF = \angle DBF = B$, and $\angle MIE = \angle ECD = C$. Note that $IE = IF$ (which equal the inradius) whence, by the Law of Sines applied to triangles MFI and MEI,

$$\frac{MF}{\sin B} = \frac{FI}{\sin \angle IMF} = \frac{EI}{\sin(\pi - \angle IMF)} = \frac{ME}{\sin C}\,.$$

We can conclude that $\frac{MF}{ME} = \frac{\sin B}{\sin C} = \frac{b}{c}$ holds by applying the Law of Sines to $\triangle ABC$, thus satisfying the requirement of equation (2). Conversely, suppose that equation (2) holds, and define M' to be the point where the line ID intersects EF. Then from the previous argument applied to M', it must be the unique point on EF for which $\frac{M'F}{M'E} = \frac{b}{c}$. We conclude that $M = M'$, and the proof is complete.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3322. [2008 : 104, 106] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be nonnegative real numbers such that $a \le b \le c$, and let n be a positive integer. Prove that

$$\big(a + (n+1)b\big)\big(b + (n+2)c\big)\big(c + na\big) \;\ge\; (n+1)(n+2)(n+3)abc\,.$$

I. *Solution by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain.*

We will prove a more general result: If $0 < a_1 \le a_2 \le \cdots \le a_n$ and $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, then

$$\begin{aligned}(a_1 + \lambda_2 a_2)\,(a_2 + \lambda_3 a_3) \cdots (a_n + \lambda_1 a_1) \\ \ge\; (1+\lambda_1)(1+\lambda_2)\cdots(1+\lambda_n) a_1 a_2 \cdots a_n\,.\end{aligned} \tag{1}$$

To prove this, we will use Karamata's inequality applied to the concave function $f(x) = \log x$.

Let r be such that

$$a_r + \lambda_{r+1}a_{r+1} \;\geq\; a_n + \lambda_1 a_1 \;\geq\; a_{r-1} + \lambda_r a_r \,.$$

Then, it is not difficult to see that

$$\begin{aligned} &(a_n + \lambda_n a_n, a_{n-1} + \lambda_{n-1}a_{n-1}, \ldots, a_1 + \lambda_1 a_1) \\ &\quad \succ \;\; (a_{n-1} + \lambda_n a_n, \ldots, a_r + \lambda_{r+1}a_{r+1}; a_n + \lambda_1 a_1, \ldots, a_1 + \lambda_2 a_2) \,. \end{aligned}$$

By Karamata's inequality,

$$\begin{aligned} &\log(a_1 + \lambda_2 a_2) + \log(a_2 + \lambda_3 a_3) + \cdots + \log(a_n + \lambda_1 a_1) \\ &\quad \geq \;\; \log(a_1 + \lambda_1 a_1) + \log(a_2 + \lambda_2 a_2) + \cdots + \log(a_n + \lambda_n a_n) \end{aligned}$$

and exponentiating yields inequality (1).

II. *Solution by Titu Zvonaru, Comăneşti, Romania.*

By the AM–GM Inequality we have

$$\frac{a + (n+1)b}{n+2} \;\geq\; a^{\frac{1}{n+2}} b^{\frac{n+1}{n+2}}\,; \qquad \frac{b + (n+2)c}{n+3} \;\geq\; b^{\frac{1}{n+3}} c^{\frac{n+2}{n+3}}\,;$$

$$\frac{c + na}{n+1} \;\geq\; c^{\frac{1}{n+1}} a^{\frac{n}{n+1}} \,.$$

These three inequalities imply that

$$\big(a + (n+1)b\big)\big(b + (n+2)c\big)(c + na) \;\geq\; (n+1)(n+2)(n+3)a^p b^q c^r \,,$$

where $p = \dfrac{n^2 + 3n + 1}{(n+1)(n+2)}$, $q = \dfrac{n^2 + 5n + 5}{(n+2)(n+3)}$, and $r = \dfrac{n^2 + 4n + 5}{(n+1)(n+3)}$. Now, since $0 \leq a \leq b \leq c$ and $r = 1 + \dfrac{1}{(n+1)(n+2)} + \dfrac{1}{(n+2)(n+3)}$, we have

$$c^r \;=\; c^1\, c^{\frac{1}{(n+1)(n+2)}}\, c^{\frac{1}{(n+2)(n+3)}} \;\geq\; a^{\frac{1}{(n+1)(n+2)}}\, b^{\frac{1}{(n+2)(n+3)}}\, c \,,$$

and hence,

$$(n+1)(n+2)(n+3)a^p b^q c^r \;\geq\; (n+1)(n+2)(n+3)abc \,.$$

This proves the proposed inequality. Equality holds if and only if $a = b = c$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer. There was one incorrect solution submitted.

Several solvers noted that the inequality holds if the positive integer n is replaced by a nonnegative real number.

3323. [2008 : 104, 106] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be nonnegative real numbers with $a^2 + b^2 + c^2 = 1$. Prove that

$$\sum_{\text{cyclic}} \left(1 - 2a^2\right)(b - c)^2 \geq 0 .$$

Solution by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain, expanded by the editor.

Since $1 = a^2 + b^2 + c^2$, we have

$$\begin{aligned}\sum_{\text{cyclic}} \left(1 - 2a^2\right)(b - c)^2 &= \sum_{\text{cyclic}} \left(-a^2 + b^2 + c^2\right)(b - c)^2 \\ &= \sum_{\text{cyclic}} a^2 \left[(a - b)^2 - (b - c)^2 + (c - a)^2\right] \\ &= \sum_{\text{cyclic}} a^2 \left(2a^2 - 2ab + 2bc - 2ca\right) \\ &= 2 \sum_{\text{cyclic}} a^2 (a - b)(a - c) ,\end{aligned}$$

which is nonnegative by Schur's inequality. Equality holds if and only if one of the numbers a, b, or c is 0 and the other two are $\frac{\sqrt{2}}{2}$, or if all three of them are $\frac{\sqrt{3}}{3}$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; GEORGE TSAPAKIDIS, Agrinio, Greece; GEORGE VELISARIS, medical student, Athens, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3324. [2008 : 104, 106] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let a, b, and c be nonnegative real numbers with $a^2 + b^2 + c^2 = 1$. Prove that

$$3 - 5(ab + bc + ca) + 6abc(a + b + c) \geq 0 .$$

Essentially similar solutions by George Apostolopoulos, Messolonghi, Greece; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

We use a standard homogenization idea: Since $1 = a^2 + b^2 + c^2$, we can write the inequality to be proved as

$$3\left(a^2+b^2+c^2\right)^2 - 5(ab+bc+ca)\left(a^2+b^2+c^2\right) + 6abc(a+b+c) \ \geq\ 0\,.$$

The above inequality is equivalent to

$$3\sum_{\text{cyclic}} a^4 \ -\ 5\sum_{\text{cyclic}}\left(a^3b+ab^3\right) \ +\ 6\sum_{\text{cyclic}} a^2b^2 \ +\ \sum_{\text{cyclic}} a^2bc \ \geq\ 0\,,$$

which we group as

$$\left(\sum_{\text{cyclic}} a^4 \ +\ \sum_{\text{cyclic}} a^2bc \ -\ \sum_{\text{cyclic}} ab\left(a^2+b^2\right)\right) + \sum_{\text{cyclic}}\left[a^4+b^4-4ab\left(a^2+b^2\right)+6a^2b^2\right] \ \geq\ 0\,,$$

which reduces to

$$\sum_{\text{cyclic}} a^2(a-b)(a-c) \ +\ \sum_{\text{cyclic}} (a-b)^4 \ \geq\ 0\,.$$

The last inequality is true, because the second sum is obviously nonnegative and the first sum is nonnegative by Schur's inequality. Equality holds if and only if $a=b=c=\sqrt{3}/3$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; GEORGE VELISARIS, medical student, Athens, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer. There was one incomplete solution submitted.

3325. [2008 : 104, 106] *Proposed by Manuel Benito Muñoz, IES P.M. Sagasta, Logroño, Spain.*

Let $\sigma(n)$ denote the sum of the divisors of the natural number n.

(a) Find a natural number n such that

$$\sigma(n) + 500 \ =\ \sigma(n+2)\,.$$

(b)★ How many solutions are there to part (a)?

Solution to (a) *by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain.*

The prime power decompositions of 2005 and 2007 are $2005 = 5 \times 401$ and $2007 = 3^2 \times 223$, respectively.

Hence, $\sigma(2005) = 1 + 5 + 401 + 2005 = 2412$ and

$$\sigma(2007) = 1 + 3 + 9 + 223 + 669 + 2007 = 2912 = \sigma(2005) + 500\,.$$

Therefore, $n = 2005$ is a solution.

Also solved (part (a) *only) by MOHAMMED AASSILA, Strasbourg, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; OLIVER GEUPEL, Brühl, NRW, Germany; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer. No solution to part* (b) *was received, hence part* (b) *remains open.*

All solvers gave the solution $n = 2005$. By using Mathematica, Wagon found that $n = 2005$ is the only solution for $n \le 42213628$. However, $n = 42213629$ is a second solution. To see this, observe that the prime power decompositions of n and $n+2$ are $42213629 = 109 \times 387281$, and $42213631 = 229 \times 337 \times 547$, respectively. Hence, $\sigma(42213629) = 1 + 109 + 387281 + 42213629 = 42601020$ and

$$\begin{aligned}\sigma(42213631) &= 1 + 229 + 337 + 547 + 77173 + 125263 + 184339 + 42213631 \\ &= 42601520 \;=\; \sigma(42213629) + 500\,.\end{aligned}$$

By further searching, he also found a third solution: $n = 60992425$ for which $\sigma(n+2) = 81448780 = \sigma(n) + 500$. He remarked that these three solutions are the only ones for n up to 10^8, and stated that "$\cdots$ perhaps there are infinitely many [solutions]".

The density of a set $S \subseteq \mathbb{Z}^+$ is $\lim\limits_{n\to\infty} \frac{|S\cap[1,n]|}{n}$, provided the limit exists. If $n = 2^3 m$, m odd, then $\sigma(n+2) \equiv \sigma(n) \equiv 0 \pmod 3$, hence $\sigma(n+2) \ne \sigma(n) + 500$. Thus, $S_3 = \{n : n = 2^3 m,\ m \text{ an odd positive integer}\}$ has density $\frac{1}{16}$ yet contains no solution of the equation. For various integers m there are sets $S_m \subseteq \mathbb{Z}^+$ of positive density such that $\sigma(n+2) \equiv \sigma(n) \equiv 0 \pmod m$ for $n \in S_m$ but 500 is not divisible by m. This suggests sieving as an attempt to answer the question "how many solutions?" in terms of density, and for reducing the search space for solutions.