Geometric Constructions of Mixtilinear Incircles

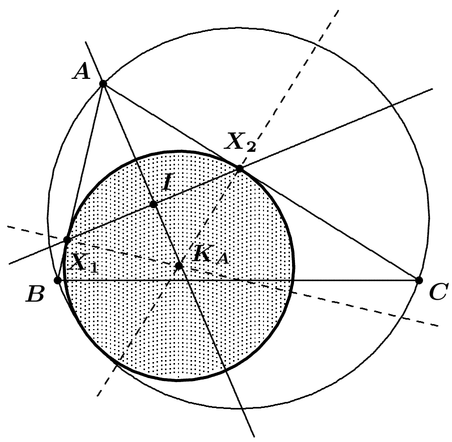
Cosmin Pohoaţă

L. Bankoff [1] introduced the term *mixtilinear incircle* of a triangle to name the three circles each tangent to two sides and to the circumcircle internally. In the same paper, Bankoff establishes the *fundamental formula* (as P. Yiu [2] refers to it) expressing the radius of a mixtilinear incircle in terms of the inradius of the triangle. More precisely, consider a triangle ABC and its mixtilinear incircle in the angle A with centre K_A and radius ρ_A. Then,

$$r \;=\; \rho_A \cos^2 \frac{\alpha}{2}\,,$$

where r is the inradius of the triangle and $\alpha = \angle BAC$. This formula leads to a first construction of the mixtilinear incircle (see [2]).

Construction 1 Denote by I the incentre of triangle ABC, and let the perpendicular through I to the internal bisector of angle A intersect the sides AB and AC at X_1 and X_2, respectively. The perpendiculars at these points to the respective sides of the triangle intersect again on the angle bisector, at the mixtilinear incentre K_A. The circle with centre K_A and passing through X_1 and X_2 is the mixtilinear incircle in angle A.

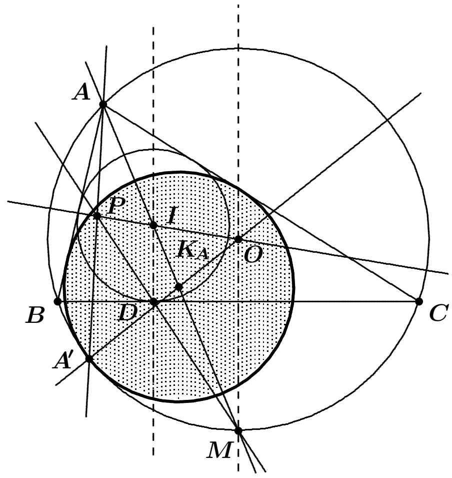


In the same paper, Yiu [2] gives an alternative construction based on the following result, whose proof we omit.

Theorem 1 (Yiu). The three lines each joining a vertex of a triangle to the point of contact of the circumcircle with the respective mixtilinear incircle are concurrent at the external center of similitude of the circumcircle and incircle.

Consider the circumcenter O, the midpoint M of the arc BC not containing the vertex A, and the tangency point D of the incircle with the side BC. Since the lines OM and ID are parallel and $OM : ID = R : r$, the intersection Q of the lines MD and OI is the external centre of similitude of the circumcircle and the incircle. This leads us to the following construction.

Construction 2 ([2]) Given a triangle ABC, let Q be the external centre of similitude of the circumcircle with centre O and incircle with centre I. Extend AQ to intersect the circumcircle at A'. The intersection of AI and $A'O$ is the centre K_A of the mixtilinear incircle in angle A.



We now give a new construction, similar at first sight to Construction 1, although it does not use Bankoff's formula or the collinearity of the incentre with the tangency points of the sides AB and AC with the mixtilinear incircle in angle A. First we will prove the following result.

Theorem 2 Let N and P be the midpoints of arcs CA and AB, not containing the vertices B and C, respectively. The reflection of A with respect to the midpoint of the segment NP lies on the line determined by the tangency points of the sides AC and AB with the mixtilinear incircle in angle A.

Proof: Let A', X_1 and X_2 be as above, and let A_1 be the reflection of the point A in the midpoint of the segment NP (see the diagram on the next page). Since A', X_1, and P are collinear and also A', X_2, and N are collinear, we have

$$\begin{aligned}\angle AA'P &= \tfrac{1}{2}\angle AA'B = \tfrac{1}{2}\angle ACB\,,\\ \angle AA'N &= \tfrac{1}{2}\angle AA'C = \tfrac{1}{2}\angle ABC\,.\end{aligned}$$

Since the quadrilateral $APA'N$ is cyclic,

$$\begin{aligned}\angle APA_1 &= \angle ANA_1 = \angle APN + \angle ANP\\ &= \angle AA'N + \angle AA'P = 90^\circ - \tfrac{1}{2}\angle BAC\,.\end{aligned}$$

On the other hand, since the triangle AX_1X_2 is isosceles,

$$\angle AX_1X_2 = \angle AX_2X_1 = 90^\circ - \frac{1}{2}\angle BAC\,.$$

Denote by A_1', A_1'' the intersections of PA_1, NA_1 with the line X_1X_2, respectively. Therefore, the quadrilaterals APX_1A_1' and ANX_2A_1'' are cyclic. Assume without loss of generality that A_1 lies on the opposite side of line CA than the vertex B does. In this case,

$$\angle APX_1 = 180^\circ - \angle AA_1'X_1\,, \quad \text{and} \quad \angle ANX_2 = \angle AA_1''X_2\,.$$

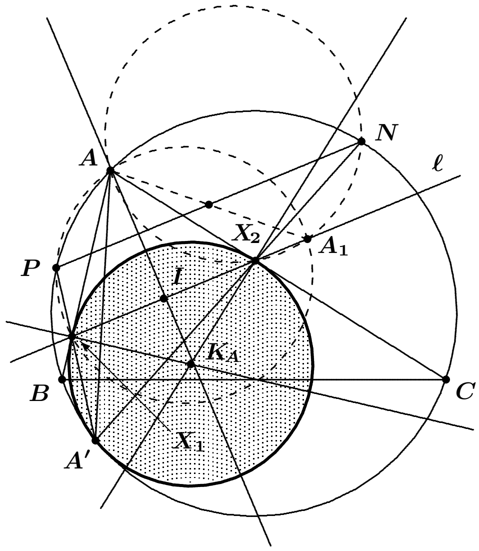
However, since the quadrilateral $APA'N$ is cyclic,

$$(180^\circ - \angle AA_1'X_1) + \angle AA_1''X_2 \;=\; \angle APA' + \angle ANA' \;=\; 180^\circ\,.$$

Hence, $\angle AA_1'X_1 = \angle AA_1''X_2$ and $A_1' = A_1''$, so A_1 lies on X_1X_2. ∎

This result leads us to our third construction:

Construction 3 Given the triangle ABC, construct the midpoints N and P of the circumcircle arcs CA and AB not containing the vertices B and C, respectively. Let A_1 be the reflection of the point A in the midpoint of the segment NP. Draw the line ℓ through A_1 perpendicular to the internal angle bisector of angle A; this line intersects AB and AC at the points X_1 and X_2, respectively. The perpendiculars at X_1 and X_2 to the sides AB and AC (respectively) intersect at K_A, the centre of the mixtilinear incircle in angle A.



We give two other related constructions involving the isogonality of the external centre of similitude of the circumcircle and incircle with the Nagel Point of the triangle. The first one is implicit in the literature.

Construction 4 In a triangle ABC construct the midpoints N and P of the circumcircle arcs CA and AB not containing the vertices C and B, respectively. Draw the tangents at N and P to the circumcircle. The second intersection of the line determined by their intersection point and the vertex A coincides with the tangency point A' of the circumcircle with the mixtilinear incircle in angle A. Hence, the lines $A'O$ and AI meet at K_A, the centre of the mixtilinear incircle in angle A.

Proof: Let M' be the antipodal point of M with respect to the circumcircle and let N_0 and P_0 be the intersection points of the tangent at M to the circumcircle with the tangents in N and P, respectively. Suppose the tangents at N and P to the circumcircle meet at the point M_0. Since the internal angle bisector AI is perpendicular to NP and to its corresponding external angle bisector AM', the lines NP and AM' are parallel. Therefore, the cyclic quadrilateral $NPAM'$ is an isosceles trapezoid, and so the lines M_0A, M_0M' are isogonal with respect to angle NM_0P. Since M_0M' is the Nagel Cevian corresponding to vertex M_0 in triangle $M_0N_0P_0$ and because triangles $M_0N_0P_0$ and ABC are homothetic, the lines M_0A and the Nagel Cevian corresponding to vertex A in triangle ABC are isogonal. ∎

We will use the following known theorem to give an argument for the isogonality of the external centre of similitude of the circumcircle and incircle with the Nagel Point of the triangle.

Theorem 3 Given a triangle ABC, let P be a point with homogeneous barycentric coordinates $(x : y : z)$.

(a) The reflection of the line AP in the internal angle bisector of vertex A intersects the line BC at the point $P_A = (0 : \frac{b^2}{y} : \frac{c^2}{z})$.

(b) If P_B and P_C are the intersection points of the reflections of the lines BP and CP in the internal angle bisectors corresponding to the vertices B and C with the lines CA and AB, respectively, then P_A, P_B, and P_C are the traces of $P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right) = \left(a^2yz : b^2zx : c^2xy\right)$, the isogonal conjugate of P. ∎

The Nagel point X_8, and the external centre of similitude of the circumcircle and incircle X_{56} have (see [3]) homogeneous barycentric coordinates

$$\begin{aligned} X_8 &= (b+c-a : c+a-b : a+b-c)\,, \\ X_{56} &= \left(\frac{a^2}{b+c-a} : \frac{b^2}{c+a-b} : \frac{c^2}{a+b-c}\right)\,. \end{aligned}$$

Hence, by Theorem 3, they are isogonal with respect to triangle ABC.

Inspired by the preceding result, we give a last simple construction based on the following theorem.

Theorem 4. Let D be the point of tangency on side BC of a triangle with its incircle. Let I_A and M be the intersection point of the internal angle bisector AI with the side BC and with the circumcircle, respectively. Then the point of tangency A' of the mixtilinear incircle in angle A lies on the circumcircle of triangle DI_AM.

Proof: Since the Nagel Cevian corresponding to vertex A and the line AA' are isogonal with respect to angle BAC, the lines $A'A_0$ and BC are parallel, where A'' is the second intersection of the Nagel Cevian through A with the circumcircle. Let A_0 be the intersection point of the parallel through A to BC with the circumcircle. Hence, the cyclic quadrilateral $AA_0A''A'$ is an isosceles trapezoid, and by symmetry the line A_0A' passes through D. On other hand, also due to the symmetry, triangles A_0CB and ABC are congruent and therefore $A_0M = AM$. Hence,

$$\angle DA'M = \angle A_0A'M = \angle A_0BC + \angle MBC = \angle C + \frac{1}{2}\angle A\,,$$

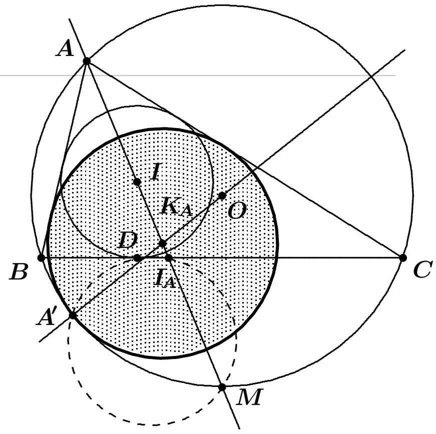
and since

$$\angle CDM = \angle CAM + \angle ACB = \angle C + \frac{1}{2}\angle A\,,$$

we conclude that the quadrilateral $DA'MI_A$ is cyclic. ∎

This being said, we can now formulate our last construction.

Construction 5 Given a triangle ABC, let D be the point of tangency of the incircle with the side BC. Let I_A and M be the intersection points of the internal angle bisector corresponding to vertex A with BC and with the circumcircle, respectively. The second intersection of the circumcircles of triangles DI_AM and ABC coincides with A', the tangency point of the circumcircle of ABC with the mixtilinear incircle in angle A.



References

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[2] P. Yiu, Mixtilinear Incircles, *Amer. Math. Monthly*, Vol. 106, No. 10 (Dec., 1999), pp. 952-955.

[3] C. Kimberling, Encyclopedia of Triangle Centers, `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`

[4] K.L. Nguyen and J.C. Salazar, On Mixtilinear Incircles and Excircles, *Forum Geom.*, Vol. 6 (2006), pp. 1-16.

Cosmin Pohoaţă
Tudor Vianu National College
Bucharest, Romania RO-010014
pohoata_cosmin2000@yahoo.com