M381. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Déterminer toutes les solutions de l'équation

$$\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-6} + \frac{1}{x-7} = x^2 - 4x - 4 \,.$$

Mayhem Solutions

M338. *Proposed by the Mayhem Staff.*

Two students miscopy the quadratic equation $x^2 + bx + c = 0$ that their teacher writes on the board. Jim copies b correctly but miscopies c; his equation has roots 5 and 4. Vazz copies c correctly, but miscopies b; his equation has roots 2 and 4. What are the roots of the original equation?

Solution by Taylor Thetford, student, Lakeview High School, San Angelo, TX, USA.

The roots of the quadratic equation that Jim writes down are 5 and 4. His quadratic equation is thus $(x-5)(x-4) = x^2 - 9x + 20 = 0$. Since Jim copied b correctly, we can conclude that in the original quadratic equation, $b = -9$.

Similarly, since Vazz's roots are 2 and 4, his quadratic equation has the form $(x-2)(x-4) = x^2 - 6x + 8 = 0$. Since Vazz copied c correctly, then $c = 8$.

Thus, the original equation was $x^2 - 9x + 8 = 0$. Factoring, we obtain $(x-1)(x-8) = 0$. Therefore, the roots of the original equation are 1 and 8.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON; and TITU ZVONARU, Comănești, Romania.

M339. *Proposed by the Mayhem Staff.*

(a) Determine the number of integers between 100 and 199, inclusive, which contain *exactly* two equal digits.

(b) An integer between 1 and 999 is chosen at random, with each integer being equally likely to be chosen. What is the probability that the integer has *exactly* two equal digits?

Solutions by Taylor Thetford, student, Lakeview High School, San Angelo, TX, USA.

(a) Each of the integers in the given range can be written in the form $1xy$. There are three cases to consider.

Case 1. The first and last digits are the same. Here, we are looking for integers $1x1$ where the middle digit can take any value except 1. This yields 9 possibilities.

Case 2. The first and second digits are the same. Here, we are looking for integers $11y$ where the last digit can take any value except 1. This again yields 9 possibilities.

Case 3. The second and last digits are the same. Here, we are looking for integers $1xx$ where x is not 1. Again, there are 9 possibilities.

Adding the results from our three cases, we find that there are 27 numbers between 100 and 199, inclusive, that contain exactly two equal digits.

(b) We count the number of integers in the range 1 to 999, inclusive, that have exactly two equal digits.

First, between 1 and 99, there are 9 of these, namely, 11, 22, $\dots$, 99. Next, between 100 and 199, we have counted 27 in part (a). Using the same argument as in (a), we can show that there are 27 numbers between 200 and 299, inclusive, and for every other interval of one hundred numbers up to the range of 900 to 999.

There are therefore $9+9\cdot 27 = 252$ numbers between 1 and 999 which contain exactly two equal digits.

The probability that a randomly selected integer between 1 and 999 has exactly two equal digits is thus $\frac{252}{999} = \frac{28}{111}$.

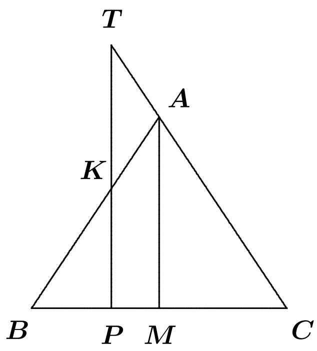
Also solved by PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON. There were 3 incorrect solutions and 1 partial solution submitted.

M340. *Proposed by the Mayhem Staff.*

Let ABC be an isosceles triangle with $AB = AC$, and let M be the mid-point of BC. Let P be any point on BM. A perpendicular is drawn to BC at P, meeting BA at K and CA extended at T. Prove that $PK + PT$ is independent of the position of P (that is, the value of $PK + PT$ is always the same, no matter where P is placed).

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Since $\triangle ABC$ is isosceles with sides AB and AC of equal length, we have $MA \perp BC$. Also, since $PT \perp BC$, then $MA || PT$.



Since $MA || PK$, then $\triangle MBA$ is similar to $\triangle PBK$ since each is right-angled and they share the angle at B. From this, we obtain $\frac{PK}{PB} = \frac{MA}{MB}$, hence $PK = \frac{PB \cdot MA}{MB}$.

Similarly, since $MA || PT$, then $\triangle CPT$ is similar to $\triangle CMA$, whence $\frac{PT}{PC} = \frac{MA}{MC}$ and so $PT = \frac{PC \cdot MA}{MC}$.

Since $MB = MC = \frac{1}{2}BC$, we can conclude that

$$\begin{aligned} PK + PT &= \frac{PB \cdot MA}{MB} + \frac{PC \cdot MA}{MC} = \frac{(PB + PC) \cdot MA}{MB} \\ &= \frac{BC \cdot MA}{MB} = 2MA\,. \end{aligned}$$

Thus, $PK + PT$ is independent of the position of P, since it depends only on the length of MA.

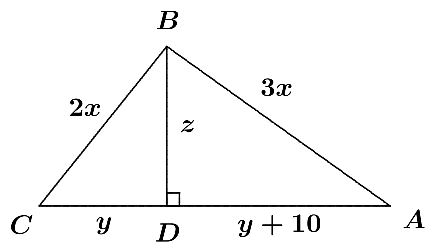
Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; and TITU ZVONARU, Comănești, Romania. There was 1 incorrect solution submitted.

M341. *Proposed by the Mayhem Staff.*

Let ABC be a right triangle with right angle at B. Sides BA and BC are in the ratio $3 : 2$. Altitude BD divides CA into two parts that differ in length by 10. What is the length of CA?

Solution by Taylor Thetford, student, Lakeview High School, San Angelo, TX, USA.

Let $2x$ and $3x$ be the lengths of CB and AB, respectively. Let y and $y + 10$ be the lengths of CD and DA, respectively. Let z be the length of BD. We wish to find $2y + 10$, which is the length of CA.



By applying the Pythagorean Theorem in $\triangle ABC$, we find that $(2x)^2 + (3x)^2 = (2y + 10)^2$ and so $13x^2 = 4y^2 + 40y + 100$.

Applying the Pythagorean Theorem to $\triangle BDC$ and $\triangle BDA$, we find that $y^2 + z^2 = 4x^2$ and $z^2 + (y + 10)^2 = 9x^2$.

Eliminating z in the last two equations gives $4x^2 - y^2 = 9x^2 - (y+10)^2$. Therefore, $5x^2 = (y + 10)^2 - y^2 = 20y + 100$ or $x^2 = 4y + 20$, and so $13x^2 = 52y + 260$.

Combining this result with $13x^2 = 4y^2 + 40y + 100$, we find that

$$\begin{aligned} 52y + 260 &= 4y^2 + 40y + 100\,; \\ 4y^2 - 12y - 160 &= 0\,; \\ y^2 - 3y - 40 &= 0\,; \\ (y + 5)(y - 8) &= 0\,. \end{aligned}$$

Since $y > 0$, then $y = 8$, and so $CA = 2y + 10 = 26$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON; and TITU ZVONARU, Comănești, Romania. There was 1 incorrect solution submitted.

M342. *Proposed by the Mayhem Staff.*

Quincy and Celine have to move 10 small boxes and 10 large boxes. The chart below indicates the time that each person takes to move each type of box.

	Celine	Quincy
small box	1 min.	3 min.
large box	6 min.	5 min.

They start moving the boxes at 9:00 am. What is the earliest time at which they can be finished moving all of the boxes?

Solution by Mayhem Staff.

Let x represent the number of small boxes and y represent the number of large boxes that Celine moves. Since there are 10 small boxes and 10 large boxes, then Quincy moves $10 - x$ small boxes and $10 - y$ large boxes.

Given the lengths of time that each takes, it takes Celine $x+6y$ minutes and it takes Quincy $3(10 - x) + 5(10 - y) = 80 - 3x - 5y$ minutes. If $x = 9$ and $y = 4$, then Celine takes 33 minutes and Quincy takes 33 minutes. We show that it cannot be done faster than this.

If Quincy and Celine finish in fewer than 33 minutes, then each takes at most 32 minutes, so the total working time is at most 64 minutes, so $x + 6y + (80 - 3x - 5y) = 80 - 2x + y \le 64$ or $2x - y \ge 16$.

Since x and y are nonnegative integers and each is less than 10, then the possible pairs (x, y) that satisfy this inequality are $(8, 0)$, $(9, 0)$, $(9, 1)$, $(9, 2)$, $(10, 0)$, $(10, 1)$, $(10, 2)$, $(10, 3)$, and $(10, 4)$.

Since we want each of Celine's time and Quincy's time to be at most 32 minutes, then we need $x + 6y \leq 32$ and $80 - 3x - 5y \leq 32$. The first inequality eliminates the pair $(10, 4)$ from the list of possible pairs. The second inequality simplifies to $3x + 5y \geq 48$; none of the remaining pairs satisfy this inequality.

Thus, none of these possibilities take any less time than 33 minutes. Therefore, the earliest possible finishing time is 9:33 a.m.

There were 4 incorrect and 3 incomplete solutions submitted.

*An expanded treatment of a similar problem appeared in the Problem of the Month column in **CRUX with MAYHEM**, volume 34, number 2.*

M343. *Proposed by the Mayhem Staff.*

The Fibonacci numbers are defined by $f_1 = f_2 = 1$ and, for $n \geq 2$, by $f_{n+1} = f_n + f_{n-1}$. The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, $\ldots$. Find the sum of the first 100 even Fibonacci numbers.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Since $f_1 = f_2 = 1$, $f_3 = 2$, and $f_n = f_{n-1} + f_{n-2}$, then f_m is even if and only if m is a multiple of 3. (This is because the parities of the terms will form the pattern Odd, Odd, Even, Odd, Odd, Even, and so on.)

If $S_n = \sum\limits_{k=1}^{n} f_{3k}$, then

$$S_n \;=\; \frac{1}{2}\sum_{k=1}^{n}(f_{3k}+f_{3k}) \;=\; \frac{1}{2}\sum_{k=1}^{n}\big((f_{3k-2}+f_{3k-1})+f_{3k}\big) \;=\; \frac{1}{2}\sum_{k=1}^{3n} f_k\,. \quad (1)$$

Next, we have $f_1 = f_3 - f_2$, and $f_2 = f_4 - f_3$, and also $f_3 = f_5 - f_4$, and so on until $f_{r-1} = f_{r+1} - f_r$ and $f_r = f_{r+2} - f_{r+1}$.

Since the right side of the sum of the n equations above "telescopes", it follows that

$$\sum_{k=1}^{r} f_k \;=\; f_{r+2} - f_2 \;=\; f_{r+2} - 1\,. \quad (2)$$

From (1) and (2), we find that $S_n = \frac{1}{2}(f_{3n+2} - 1)$. In our particular case, $S_{100} = \frac{1}{2}(f_{302} - 1)$. Maple computes the value of S_{100} to be exactly 290905784918002003245752779317049533129517076702883498623284700.

For the record, by Binet's formula for Fibonacci numbers we have that $f_m = \dfrac{1}{\sqrt{5}}\left(\alpha^m - \beta^m\right)$, where $\alpha = \dfrac{1}{2}(1+\sqrt{5})$ and $\beta = \dfrac{1}{2}(1+\sqrt{5})$. Hence the required sum is also given by $S_{100} = \dfrac{1}{2\sqrt{5}}\left(\alpha^{302} - \beta^{302}\right) - \dfrac{1}{2}$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; DIVYANSHU RANJAN, Delhi, India; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; and TITU ZVONARU, Comănești, Romania.