Velocity Analysis: an Approach to Solving Geometry Problems

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1 Introduction

We introduce velocity analysis for solving otherwise complex geometry problems, then we give two examples of the use of this method: (1) to give a very brief proof of the optical property of the ellipse, and (2) to find the length of a logarithmic spiral. At the end of this note we leave some problems for the reader; the solutions will be very brief if velocity analysis is used.

2 Velocity Analysis

In analytic geometry, a curve is a set of points whose coordinates (x, y) satisfy a certain formula. For example, the set of points $\{(x, y) : x^2 + y^2 = R^2\}$ is a circle or radius R.

As we all know, the trace of a moving point is a curve. In the analytic notion of a curve above, we mainly pay attention to the *position* of a moving point, rather than its velocity. In fact, either the position or the velocity of a moving point can describe the course of its motion, and in some cases it is more convenient to study the velocity rather than the position. If instead of studying the coordinates of a moving point we study its velocity, then it is very simple for us to deduce certain geometric properties of the traced curve without engaging in complex mathematics, especially when seeking the tangent or arc length. (As we know, the velocity vector of a moving point gives a tangent to the curve traced by the point.)

Now we introduce the basic idea of velocity analysis in solving geometry problems.

A point P is moving under a certain restriction on its velocity $\overrightarrow{V}$. For example, if $\overrightarrow{V} \cdot \overrightarrow{OP} \equiv 0$ (here $\overrightarrow{OP}$ is the position vector of P), then obviously the point P traces out a circle whose centre is O. That is to say, we find an equivalent way of defining the circle by studying the velocity rather than the coordinates of a moving point.

When studying velocity, usually it is more convenient to study components. In our example of the circle, we break the velocity of P into two directions: along $\overrightarrow{OP}$ and orthogonal to $\overrightarrow{OP}$. Name the two components $\overrightarrow{V_1}$

and $\overrightarrow{V_2}$ respectively; if the component $\overrightarrow{V_1}$ along $\overrightarrow{OP}$ satisfies $\overrightarrow{V_1} \equiv 0$, then the point P traces out a circle.

This is the basic idea of velocity analysis: if we decompose the velocity of a moving point into two appropriate directions, giving certain restrictions to the components of the velocity, the trace of the moving point will become a certain curve. In this setting it is very easy to analyse the tangent to the curve, for we already know the restriction on the velocity of the point.

Example 1 The optical property of the ellipse. To show the power of the vector analysis approach, we use it to solve this classical geometry problem. In ancient times people found that conic sections have very special and beautiful optical properties. One example is this: if a ray of light leaves one focus of an ellipse and strikes the ellipse, it will be reflected to the other focus of the ellipse (see Figure 1).

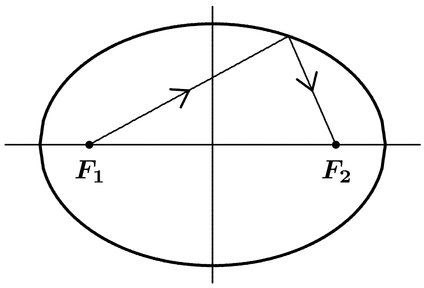


Figure 1

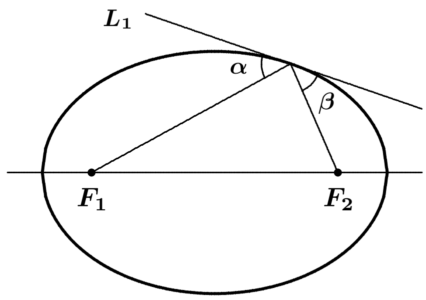


Figure 2

Using calculus to prove this property would be complicated, so we use the approach of analysing the velocity of a moving point to give a much shorter proof. As in Figure 2, to prove the optical property of the ellipse, we need to prove that $\alpha = \beta$, which is enough to satisfy the Law of Reflection. Here L_1 is tangent to the ellipse.

We usually define an ellipse like this: two points F_1 and F_2 in the plane are fixed. A point P moves so that $|\overrightarrow{PF_1}| + |\overrightarrow{PF_2}|$ is always a constant. We call the trace of P an ellipse. In accordance with the idea of velocity analysis, we describe an ellipse as follows:

Definition. Two points F_1, F_2 in the plane are fixed (see Figure 3). A moving point, P, has velocity vector $\overrightarrow{V}$. Let $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ be the components of $\overrightarrow{V}$ towards F_1 and away from F_2, respectively. If $|\overrightarrow{V_1}| \equiv |\overrightarrow{V_2}|$, then the trace of P is an ellipse.

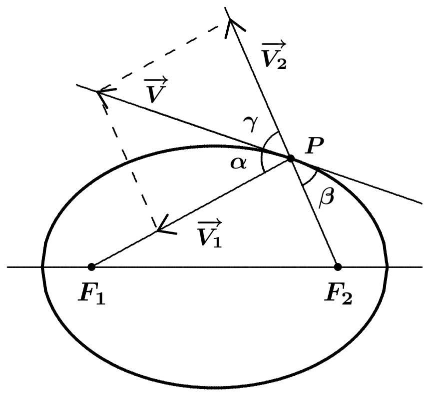


Figure 3

It is obvious that the two definitions are equivalent. For when P moves towards F_1 a short distance, it must then move away from F_2 by that same distance to ensure that $|\overrightarrow{PF_1}| + |\overrightarrow{PF_2}|$ is always a constant.

Now $\overrightarrow{V}$ is a tangent vector of the ellipse. Since $|\overrightarrow{V_1}| \equiv |\overrightarrow{V_2}|$, then $\alpha = \gamma$. Since $\beta = \gamma$, we also have $\alpha = \beta$. That is all.

Example 2 The length of the logarithmic spiral. As another example, we find the length of the logarithmic spiral $\rho = \rho_0 e^{a\theta}$ between $\rho = \rho_1$ and $\rho = \rho_2$. Usually we cannot figure out this problem without using a lot of calculus, so we introduce a physical model.

As in Figure 4, three points A, B, and C are each at a vertex of an equilateral triangle of side $\sqrt{3}\rho_0$. Point A moves towards B, B moves towards C, and C moves towards A. The speed of each point is s. Where will the three points meet and how far will they travel before meeting?

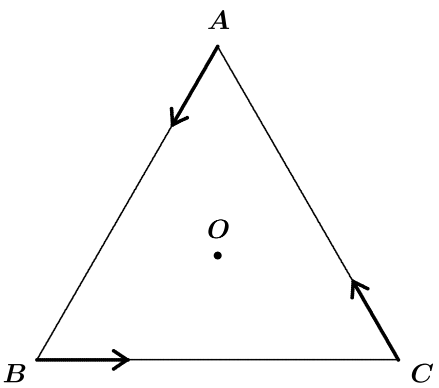


Figure 4

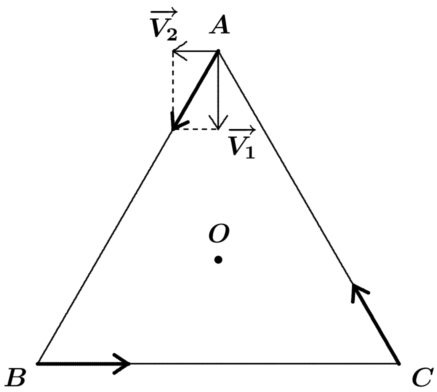


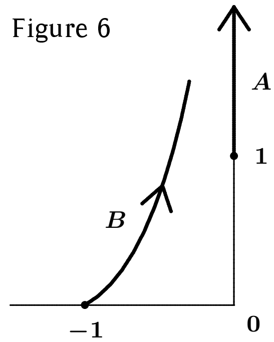
Figure 5

The solution of this problem is easy: obviously the three points will meet at the point O, which is the centre of the triangle. We decompose the velocity $\overrightarrow{V}$ of A at each instant into two components: one towards the point O and the other perpendicular to $\overrightarrow{AO}$, and we call these two components $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$, respectively. We have that $|\overrightarrow{V_1}| = |\overrightarrow{V}|\cos 30^\circ = \frac{\sqrt{3}}{2}s$ (see Figure 5), so the time until they meet is $t = \frac{\rho_0}{|\overrightarrow{V_1}|} = \frac{\rho_0}{(\sqrt{3}/2)s}$ and the distance traveled is $d = st = \frac{2\rho_0}{\sqrt{3}}$.

Now we consider the curve traced by A. We use a polar coordinate system with the origin at O and with the vector $\overrightarrow{V_2}$ pointing in the direction of increasing θ. At each instant, $|\overrightarrow{V_2}| = |\overrightarrow{V}|\sin 30^\circ = |\overrightarrow{V_1}|\tan 30^\circ$. Since $|\overrightarrow{V_2}| = \frac{\rho d\theta}{dt}$ and $|\overrightarrow{V_1}| = -\frac{d\rho}{dt}$, we have $d\rho = a\rho d\theta$, the differential equation of the spiral $\rho = \rho_0 e^{a\theta}$ with $a = \cot 150^\circ = -\sqrt{3}$. Thus, the length of the curve from $\rho = \rho_1$ to $\rho = \rho_2$ is $\frac{|\rho_1 - \rho_2|}{|\overrightarrow{V_1}|} \cdot s = \frac{2}{\sqrt{3}}|\rho_1 - \rho_2|$. We leave the problem of finding a suitable physical model for other values of a to the reader.

We leave three more problems for the reader to solve. They could be solved by using analytic geometry and calculus, but are more conveniently solved using the approach of velocity analysis.

Figure 6



1. The pursuant trajectory problem. As in Figure 6, suppose that an object A starts from the point $(0, 1)$, and moves with a constant speed s in the direction of the positive y–axis. At the same time another object B starts from the point $(-1, 0)$, and moves with speed $2s$ and always in the direction of the object A. When will the objects meet?

2. The optical property of a hyperbola. If a light ray leaves one focus of a hyperbola and strikes the hyperbola, then the (reverse) extended line of its reflection will pass through the other focus of the hyperbola (see Figure 7).

3. The optical property of a parabola. If a light ray leaving the focus is reflected in the parabola, then its reflection is parallel to the axis of symmetry (see Figure 8).

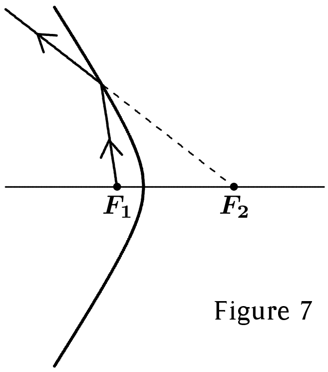


Figure 7

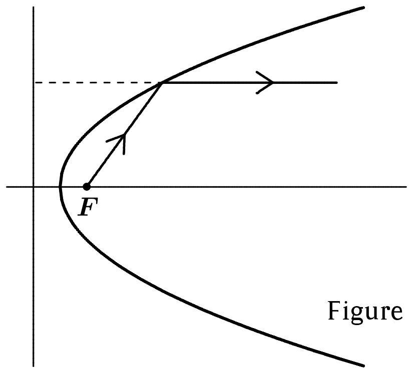


Figure 8

References

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[2] R.T. Coffman and C.S. Ogilvy, The "Reflection Property" of the Conics, *Mathematics Magazine*, Vol. 36, No. 1 (Jan., 1963), pp. 11-12.

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