THE OLYMPIAD CORNER

No. 275

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The year has flown by, and it has brought many changes to Crux and to the *Corner*. Of course it has been overshadowed by the sudden and untimely loss of a great friend and a devoted colleague, Jim Totten, mid-way through the transition to a new Editor-in-Chief. I think Vazz Linek has done a wonderful job of stepping in and keeping the journal on track with only an understandable slowing of the production pace in the interim.

Readers will have noticed the announcement at the end of the December *Corner* that Joanne Canape, who has transformed my scribbles into a high quality tex file for many years, has decided that two decades is enough. As I customarily begin the year by thanking all those who contributed to the *Corner* in the last year, I would be very remiss not to lead off with sincere thanks to Joanne.

It is also appropriate to thank those who submitted problem sets for our use as well as a special thanks to the dedicated readers who furnish their nice solutions which we use. Hoping, as always, that I've not missed someone, here is the list for the 2008 members of the *Corner*.

Arkady Alt
Miguel Amengual Covas
Jean-Claude Andrieux
Houda Anoun
Ricardo Barroso Campos
Michel Bataille
José Luis Díaz-Barrero
J. Chris Fisher
Kipp Johnson
Geoffrey A. Kandall
Ioannis Katsikis
R. Laumen
Salem Malikic
Pavlos Maragoudakis
Robert Morewood
Andrea Munaro
Vedula N. Murty
Felix Recio
Xavier Ros
D.J. Smeenk
Babis Stergiou
Daniel Tsai
Panos E. Tsaoussoglou
George Tsapakidis
Jan Verster
Edward T.H. Wang
Luyan Zhong-Qiao
Li Zhou
Titu Zvonaru

Our apologies to Svetoslav Savchev for the misspelling of his name in the December 2008 Olympiad.

For your problem solving pleasure in the new year we start off with the problems of the German Mathematical Olympiad, Final Round, 2006. My thanks go to Robert Morewood, Canadian Team Leader to the 47$^{\text{th}}$ IMO in Slovenia 2006, for collecting them for our use.

German Mathematical Olympiad
Final Round, Grades 12–13
Munich, April 29 – May 2, 2006

First Day

1. Determine all positive integers n for which the number

$$z_n = \underbrace{101\cdots101}_{2n+1 \text{ digits}}$$

is a prime.

2. Five points are on the surface of a sphere of radius 1. Let $a_{\min}$ denote the smallest distance (measured along a straight line in space) between any two of these points. What is the maximum value for $a_{\min}$, taken over all arrangements of the five points?

3. Find all positive integers n for which the numbers $1, 2, 3, \ldots, 2n$ can be coloured with n colours in such a way that every colour appears twice and every number $1, 2, 3, \ldots, n$ appears exactly once as the difference of two numbers with the same color.

Second Day

4. Let D be a point inside the triangle ABC such that $AC - AD \geq 1$ and $BC - BD \geq 1$. Prove that $EC - ED \geq 1$ for any point E on the side AB.

5. Let x be a nonzero real number satisfying the equation $ax^2 + bx + c = 0$. Furthermore, let a, b, and c be integers satisfying $|a| + |b| + |c| > 1$. Prove that

$$|x| \geq \frac{1}{|a| + |b| + |c| - 1}.$$

6. Let a circle through B and C of a triangle ABC intersect AB and AC in Y and Z, respectively. Let P be the intersection of BZ and CY, and let X be the intersection of AP and BC. Let M be the point that is distinct from X and on the intersection of the circumcircle of the triangle XYZ with BC. Prove that M is the midpoint of BC

Our second problem set for this number is a set of selected problems from the Thai Mathematical Olympiad Examinations 2005. Again, thanks go to Robert Morewood, team leader to the 47th IMO in Slovenia 2006, for collecting them for the *Corner*.

Thai Mathematical Olympiad Examinations 2005
Selected Problems

1. Let $P(x)$, $Q(x)$, and $R(x)$ be polynomials satisfying

$$2xP(x^3) \ + \ Q(-x-x^2) \ = \ (1+x+x^2)\,R(x)\,.$$

Show that $x-1$ is a factor of $P(x)-Q(x)$.

2. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y+f(xy)) \ = \ f(f(x+y)) \ + \ xy$$

for all x, $y\in\mathbb{R}$.

3. Let a, b, and c be positive real numbers. Prove that

$$1 \ + \ \frac{3}{ab+bc+ca} \ \geq \ \frac{6}{a+b+c}\,.$$

4. Let n be a positive integer. Prove that $n(n+1)(n+2)$ is not a perfect square.

5. Find the least positive integer n such that $2549 \mid \left(n^{2545}-2541\right)$.

6. Do there exist positive integers x, y, and z such that

$$2548^x \ + \ (-2005)^y \ = \ (-543)^z\,?$$

7. Show that there exist positive integers m and n such that $\gcd(m,n)=1$ and $2549 \mid ((25\cdot 49)^m + 25^n - 2\cdot 49^n)$.

8. The median AM of a triangle ABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y, respectively. The lines AX and AY intersect BC at P and Q. Prove that $BC=CQ$. (Shortlist 2005)

9. Let ABC be an acute-angled triangle with $AB\neq AC$, let H be its orthocentre and M the midpoint of BC. Points D on AB and E on AC are such that $AE=AD$ and D, H, and E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE. (Shortlist 2005)

10. Assume ABC is an isosceles triangle with $AB=AC$. Suppose that P is a point on the extension of side BC. X and Y are points on lines AB and AC such that $PX \| AC$ and $PY \| AB$. Let T be the midpoint of arc BC. Prove that $PT\perp XY$. (Iran 2004)

As a third set of problems we give the 46th Ukrainian Mathematical Olympiad Final Round 2006 - 11th form problems. Again, thanks go to Robert Morewood, team leader to the 47th IMO in Slovenia 2006, for collecting them for our use.

46th Ukrainian Mathematical Olympiad 2006
Final Round
11th Form

1. (V.V. Plakhotnyk) Prove that for any rational numbers a and b the graph of the function

$$f(x) \;=\; x^3 - 6abx - 2a^3 - 4b^3\,, \quad x \in \mathbb{R}$$

has exactly one point in common with the x-axis.

2. (O.A. Sarana) A circle is divided into 2006 equal arcs by 2006 points. Baron Munchausen claims that he can construct a closed polygonal curve with the set of vertices consisting of these 2006 points such that amongst its 2006 edges there are no two which are parallel to each other. Is his claim true or false?

3. (T.M. Mitelman)

(a) Prove that for any rational number $\alpha \in (0,1)$ there exists an infinite set of real numbers that satisfy the equation $\{x\lfloor x\{x\}\rfloor\} = \alpha$ and any two of them have the same fractional part. (The fractional part of a real number a is given by $\{a\} = a - \lfloor a\rfloor$, where $\lfloor a\rfloor$ is its integer part, that is, the greatest integer that does not exceed a.)

(b) Prove that for any rational number $\alpha \in (0,1)$ there exists an infinite set of real numbers that satisfy the equation $\{x\lfloor x\{x\}\rfloor\} = \alpha$ and any two of them have *different* fractional parts.

4. (V.A. Yasinskiy) Two circles ω_1 and ω_2 intersect each other at two distinct points A and B. The tangent line of the circle ω_1 at the point A and the tangent line of the circle ω_2 at the point B meet at point C. The first of these two lines intersects the circle ω_2 for the second time at point $T \neq A$. The point X (distinct from A and B) is on the circle ω_1, and the line XA intersects the circle ω_2 for the second time at point Y (distinct from A). The lines YB and XC meet at point Z. Prove that TZ is parallel to XY.

5. (O.O. Kurchenko) Prove that for any real numbers x and y

$$|\cos x| \;+\; |\cos y| \;+\; |\cos(x+y)| \;\geq\; 1\,.$$

6. (T.M. Mitelman) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^3 + y^3\right) \;=\; x^2 f(x) \;+\; y f(y^2)$$

for all real numbers x and y.

7. (V.A. Yasinskiy) A point M lies inside a cube $ABCDA_1B_1C_1D_1$. Points A', B', C', D', A_1', B_1', C_1', and D_1' belong to the rays MA, MB, MC, MD, MA_1, MB_1, BC_1, and MD_1 respectively. Prove that if the polyhedron $A'B'C'D'A_1'B_1'C_1'D_1'$ is a parallelepiped (that is, all of its faces are parallelograms), then it is a cube.

8. (V.A. Yasinskiy) There are $n \geq 3$ soldiers in captain Petrenko's squad, no two of the same height. The captain orders them to stand single-file (not necessarily sorted by height). A "wave" is any subsequence of (not necessarily next to each other) soldiers in this line such that the first (leftmost) soldier in the wave is taller than the second soldier in it, but the second soldier in it is shorter than the third one, who is in turn taller than the fourth one, and so on. (For example, if $n = 9$, the soldiers are enumerated by height, and the captain lines them up as 9, 3, 5, 7, 1, 2, 6, 4, 8 then a longest wave for this line-up is 9, 3, 7, 1, 6, 4, 8. However, if the captain lines them up as 1, 2, 3, 4, 5, 6, 7, 8, 9, then a longest wave consists of (any) one soldier.) For every n, consider the number of possible lines with the longest waves of even lengths and the number of possible lines with the longest waves of odd lengths. Which of these numbers is bigger?

Continuing with problems for readers to solve we give the Czech-Polish-Slovak Mathematics Competition written on June 26-28, 2006 at Žilina, Slovakia. Thanks again go to Robert Morewood, Canadian team leader to the 47$^{\text{th}}$ IMO in Slovenia 2006, for collecting them for our use.

Czech-Polish-Slovak Mathematics Competition 2006

1. Five distinct points A, B, C, D, and E lie in this order on a circle of radius r and satisfy $AC = BD = CE = r$. Prove that the orthocentres of the triangles ACD, BCD, and BCE are the vertices of a right-angled triangle.

2. There are n children sitting at a round table. Erika is the oldest among them and she has n candies. No other child has any candy. Erika distributes the candies as follows. In every round, all the children with at least two candies show their hands. Erika chooses one of them and he/she gives one candy to each of the children sitting next to him/her. (So in the first round Erika must choose herself to begin the distribution.) For which $n \geq 3$ is it possible to redistribute the candies so that each child has exactly one candy?

3. The sum of four real numbers is 9 and the sum of their squares is 21. Prove that these four numbers can be labelled as a, b, c, and d so that the inequality $ab - cd \geq 2$ holds.

4. Prove that for every positive integer k there is a positive integer n such that the decimal representation of 2^n has a block of exactly k consecutive zeros, that is, $2^n = \cdots a00\cdots 0b\cdots$, where a and b are nonzero digits with k zeros between them.

5. Find the number of integer sequences $(a_n)_{n=1}^{\infty}$ such that $a_n \neq -1$ and

$$a_{n+2} = \frac{a_n + 2006}{a_{n+1} + 1}$$

for every positive integer n.

6. Is there a convex pentagon $A_1A_2\dots A_5$ such that for each i the lines A_iA_{i+3} and $A_{i+1}A_{i+2}$ intersect in B_i and the points B_1, B_2, $\dots$, B_5 are collinear? (By convention $A_6 = A_1$, $A_7 = A_2$, and $A_8 = A_3$.)

Our final problem set for this issue is the XXI Olimpiadi Italiano della Matematica, Cesenatico, written 5 May 2006. Thanks again go to Robert Morewood, Canadian team leader to the 47$^{\text{th}}$ IMO in Slovenia, for collecting them for our use.

XXI Olimpiadi Italiano della Matematica
Cesenatico
May 5, 2006

1. Rose and Savino play a game with a deck of traditional Neapolitan playing cards which consists of 40 cards of four different suits, numbered 1 to 10. At the start each player has 20 cards. Taking turns, one shows a card on the table. Whenever some cards on the table add to exactly 15, these are then removed from the game (if the sum 15 can be obtained in more than one way, the player who last moved decides which cards adding to 15 to remove). At the end of the game only one card, a 9, is left on the table. Savino holds two cards numbered 3 and 5, and Rose holds one card. What is the number of Rose's card?

2. Find all values of m, n, and p such that

$$p^n + 144 = m^2,$$

where m and n are positive integers and p is a prime number.

3. Let A and B be two points on a circle Γ such that AB is not a diameter. Let P be a point on Γ different from A and B, and let H be the orthocentre of the triangle ABP. Find the locus of H as P varies over all points of Γ different from A and B.

4. On an infinite chessboard all the positive integers are written in ascending order along a spiral, starting from 1 and proceeding anticlockwise; a portion of the chessboard is shown in the figure.

17	16	15	14	13
18	5	4	3	12
19	6	1	2	11
20	7	8	9	10
21	22	23	24	25

A "right half-line" of the chessboard is the set of squares given by a square C and by all squares in the same row as C and to the right of C.

(a) Prove that there exists a right half-line none of whose squares contains a multiple of 3.

(b) Determine if there exist infinitely many pairwise disjoint right half-lines none of whose squares contains a multiple of 3.

5. Consider the inequality

$$(x_1 + \cdots + x_n)^2 \;\geq\; 4(x_1x_2 + x_2x_3 + \cdots + x_nx_1)\,.$$

(a) Determine for which $n \geq 3$ the inequality holds true for all possible choices of positive real numbers x_1, x_2, $\dots$, x_n.

(b) Determine for which $n \geq 3$ the inequality holds true for all possible choices of *any* real numbers x_1, x_2, $\dots$, x_n.

6. Albert and Barbara play a game. At the start there are some piles of coins on the table, not all necessarily with the same number of coins. The players move in turn and Albert starts. At each turn a player may either take a coin from a pile ***or*** divide a pile into two piles with each pile containing at least one coin (a player may exercise only one of these options).

The one who takes the last coin wins the game. In terms of the number of piles and the number of coins in each pile at the start, determine which of the players has a winning strategy.

Now we turn to our file of solutions from the readers to problems from the March 2008 number of the *Corner* and the Estonian IMO Selection Contest 2004-2005, given at [2008 : 79-80].

3. Find all pairs (x, y) of positive integers satisfying $(x + y)^x = x^y$.

Solution by Konstantine Zelator, University of Toledo, Toledo, OH, USA.

We show that there are exactly two such pairs, $(x, y) = (2, 6)$, $(3, 6)$. We will make use of two basic facts from elementary number theory.

(a) If a, $b \in \mathbb{Z}$ and $\gcd(a, b) = 1$, then $\gcd(a^m, b^n) = 1$ for any positive integers m and n.

(b) If $a|b$, a is a positive integer, $b \in \mathbb{Z}$, and $\gcd(a, b) = 1$, then $a = 1$.

Let x and y be positive integers that satisfy $(x + y)^x = x^y$. Write $d = \gcd(x, y)$ and let $x = dx_1$, $y = dy_1$, where x_1 and y_1 are positive integers that are relatively prime. Substituting for x and y yields

$$d^{x_1 d} \cdot (x_1 + y_1)^{x_1 d} \;=\; x_1^{y_1 d} \cdot d^{y_1 d}\,. \tag{1}$$

We make cases by comparing the sizes of x_1 and y_1.

Case 1. Suppose that $x_1 = y_1$. Since $\gcd(x_1, y_1) = 1$, we have $x_1 = y_1 = 1$. Thus, equation (1) becomes $d^d \cdot 2^d = d^d$, which is impossible since $d \geq 1$.

Case 2. Suppose that $y_1 < x_1$. Then $x_1 - y_1$ is a positive integer and from equation (1) we obtain

$$d^{d(x_1 - y_1)} \cdot (x_1 + y_1)^{x_1 d} \;=\; x_1^{y_1 d}\,. \tag{2}$$

Since $\gcd(x_1, y_1) = 1$ it follows that $\gcd(x_1 + y_1, x_1) = 1$. By (a) above, we have $\gcd\big((x_1 + y_1)^{x_1 d}, x_1^{y_1 d}\big) = 1$. However, by equation (2), the positive integer $(x_1 + y_1)^{x_1 d}$ is a divisor of $x_1^{y_1 d}$. Since these two integers are relatively prime, it follows by (b) that $(x_1 + y_1)^{x_1 d} = 1$, which is impossible since $x_1 + y_1 \geq 2$ and $x_1 \cdot d \geq 1$.

Case 3. Suppose that $x_1 < y_1$. From (1) we obtain

$$(x_1 + y_1)^{x_1 d} \;=\; x_1^{y_1 d} \cdot d^{d(y_1 - x_1)}\,. \tag{3}$$

Since $\gcd(x_1, y_1) = 1$ we have $\gcd\big((x_1 + y_1)^{x_1 d}, x_1^{y_1 d}\big) = 1$ and from equation (3) we see that $x_1^{y_1 d}$ is a divisor of $(x_1 + y_1)^{x_1 d}$, which implies that $x_1^{y_1 d} = 1$. Since $y_1 d$ is a positive integer this means that $x_1 = 1$. Going back to equation (3) we see that $(1 + y_1)^d = d^{d(y_1 - 1)}$, hence

$$1 + y_1 \;=\; d^{y_1 - 1}\,. \tag{4}$$

Note that $d \neq 1$; otherwise equation (4) becomes $1 + y_1 = 1$, contrary to the fact that y_1 is a positive integer. Thus, $d \geq 2$. Since $y_1 = 1$ does not satisfy equation (4), we also have $y_1 \geq 2$. Setting $k = y_1 - 1$ equation (4) then becomes $k + 2 = d^k$. By Induction (or the Binomial Theorem) we obtain $2^k > k + 2$ for all integers $k \geq 3$. Since $d^k \geq 2^k$, it follows from $k + 2 = d^k$ that $k = 1$ or $k = 2$.

For $k = 2$ we have $4 = d^2$, hence $d = 2$. From $2 = k = y_1 - 1$ we then have $y_1 = 3$. Recall that $x_1 = 1$. Going back, we have $x = x_1 d = 1 \cdot 2 = 2$ and $y = y_1 d = 3 \cdot 2 = 6$. This is the solution $(x, y) = (2, 6)$.

Similarly, for $k = 1$ we have $d = 3$. Then $y_1 = k + 1 = 2$ and since $x_1 = 1$ we obtain $x = dx_1 = 3 \cdot 1 = 3$ and $y = dy_1 = 3 \cdot 2 = 6$. This is the other solution $(x, y) = (3, 6)$.

4. Find all pairs (a, b) of real numbers such that all roots of the polynomials $6x^2 - 24x - 4a$ and $x^3 + ax^2 + bx - 8$ are non-negative real numbers.

Solution by Titu Zvonaru, Comăneşti, Romania.

Let β_1, β_2, and β_3 be the roots of the polynomial $x^3 + ax^2 + bx - 8$, so that $x^3 + ax^2 + bx - 8 = (x - \beta_1)(x - \beta_2)(x - \beta_3)$. Comparing coefficients yields $\beta_1 + \beta_2 + \beta_3 = -a$ and $\beta_1\beta_2\beta_3 = 8$. Since β_1, β_2, and β_3 are nonnegative real numbers, by the AM-GM Inequality we have

$$\beta_1 + \beta_2 + \beta_3 \geq 3\sqrt[3]{\beta_1\beta_2\beta_3},$$

hence $-a \geq 6$ or $a \leq -6$. The equation $6x^2 - 24x - 4a = 0$ has real roots if and only if $24^2 - 4 \cdot (-4a) \cdot 6 \geq 0$, which implies $24^2 + 24 \cdot 4a \geq 0$ and hence $a \geq -6$. Therefore, $a = -6$.

Now we have $\beta_1 + \beta_2 + \beta_3 = 6 = 3\sqrt[3]{\beta_1\beta_2\beta_3}$, from which it follows that $\beta_1 = \beta_2 = \beta_3 = 2$ and $b = 12$. Thus, the only pair satisfying the condition is $(a, b) = (-6, 12)$.

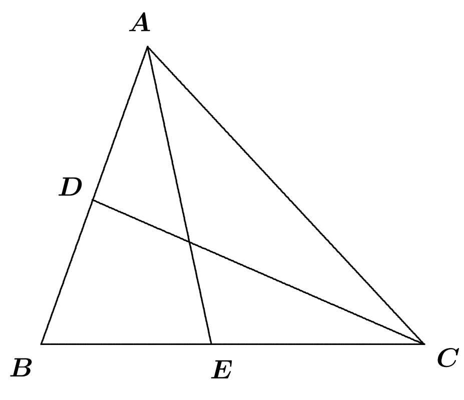
Next we turn to a solution to a problem of the Trentième Olympiad Mathématique Belge Maxi Finale, Mercredi 20 avril 2005 given at [2008 : 80].

3. Dans le triangle ABC, les droites AE et CD sont les bissectrices intérieures des angles $\angle BAC$ et $\angle ACB$ respectivement ; E appartient à BC et D appartient à AB. Pour quelles amplitudes de l'angle $\angle ABC$ a-t-on certainement

(a) $|AD| + |EC| = |AC|$? (b) $|AD| + |EC| > |AC|$?
(c) $|AD| + |EC| < |AC|$?

Solution by Titu Zvonaru, Comăneşti, Romania

As usual let $a = BC$, $b = CA$, and $c = AB$. The first equation below is the Angle Bisector Theorem; the following equations are equivalent to it:



$$\frac{BE}{EC} = \frac{AB}{AC};$$
$$\frac{BE}{EC} = \frac{c}{b};$$
$$\frac{BE + EC}{EC} = \frac{b + c}{b};$$
$$EC = \frac{ab}{b + c}.$$

Similarly, $AD = \dfrac{bc}{a + b}$.

We have

$$\begin{aligned}|AD| + |EC| - |AC| &= \frac{bc}{a+b} + \frac{ab}{b+c} - b \\ &= b\left(\frac{c(b+c)+a(a+b)-(a+b)(b+c)}{(a+b)(b+c)}\right) .\end{aligned}$$

By the Law of Cosines $a^2+c^2-b^2 = 2ac\cos B$, so the equation above can be rewritten as

$$|AD| + |EC| - |AC| = \frac{2abc\left(\cos(B)-\frac{1}{2}\right)}{(a+b)(b+c)} .$$

Hence,

$$\begin{aligned}|AD| + |EC| = |AC| &\iff \angle ABC = 60^\circ ,\\ |AD| + |EC| > |AC| &\iff \angle ABC < 60^\circ ,\\ |AD| + |EC| < |AC| &\iff \angle ABC > 60^\circ .\end{aligned}$$

Next we turn to solutions from our readers to problems of the 2005 Vietnam Mathematical Olympiad given at [2008 : 81].

1. Find the smallest and largest values of the expression $P = x + y$, where x and y are real numbers satisfying $x - 3\sqrt{x+1} = 3\sqrt{y+2} - y$.

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give the solution of Bataille.

We show that $P_{\min} = \frac{1}{2}\left(9+3\sqrt{21}\right)$ and $P_{\max} = 9 + 3\sqrt{15}$.

First, $x = -1$ and $y = \frac{1}{2}\left(11+3\sqrt{21}\right)$ satisfy the constraint equation $x-3\sqrt{x+1} = 3\sqrt{y+2}-y$ (easily checked using $10+2\sqrt{21} = \left(\sqrt{3}+\sqrt{7}\right)^2$) and $P = \frac{1}{2}\left(9+3\sqrt{21}\right)$. Similarly, for $x = \frac{1}{2}\left(10+3\sqrt{15}\right)$, $y = \frac{1}{2}\left(8+3\sqrt{15}\right)$, we have $P = 9+3\sqrt{15}$ and the constraint is satisfied. Now, let x and y satisfy the constraint equation. Then $P = 3\sqrt{x+1} + 3\sqrt{y+2}$, so that

$$P^2 = 9\left(P + 3 + 2\sqrt{(x+1)(y+2)}\right) . \qquad (1)$$

It follows that $P \geq 0$ and $P^2 - 9P - 27 \geq 0$. Thus, P is not less than the positive solution of the quadratic $x^2 - 9x - 27 = 0$ and we deduce that $P \geq \frac{1}{2}\left(9+3\sqrt{21}\right)$. From the AM-GM Inequality and (1), we obtain

$$P^2 \leq 9(P+3+x+1+y+2) = 9(2P+6) = 18P+54 ,$$

or $P^2 - 18P - 54 \leq 0$, which implies that $P \leq 9 + 3\sqrt{15}$. The proof is complete.

4. Find all real-valued functions f defined on $\mathbb{R}$ that satisfy the identity $f\big(f(x-y)\big) = f(x)f(y) - f(x) + f(y) - xy$.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Daniel Tsai, student, Taipei American School, Taipei, Taiwan. We give the write up of Bataille.

It is readily checked that the function $f(x) = -x$ for all x is a solution. We show that there are no other solutions. Let f satisfy

$$f\big(f(x-y)\big) \;=\; f(x)f(y) \;-\; f(x) \;+\; f(y) \;-\; xy \qquad (1)$$

for all x, $y \in \mathbb{R}$ and let $a = f(0)$. Taking $x = y = 0$ in (1) gives $f(a) = a^2$ and then taking $y = x$ shows that

$$f(x)^2 \;=\; x^2 + a^2$$

for all real numbers x. In particular $f(a)^2 = 2a^2$, that is, $a^4 = 2a^2$, hence $a \in \{0,\ \sqrt{2},\ -\sqrt{2}\}$. Assume that $a = \sqrt{2}$. Then for any given x we have $f(x) = \sqrt{x^2+2}$ or $f(x) = -\sqrt{x^2+2}$ and taking $y = 0$ in (1), we obtain

$$f\big(f(x)\big) \;=\; (\sqrt{2}-1)f(x) \;+\; \sqrt{2}\,. \qquad (2)$$

Now, $f\big(\sqrt{2}\,\big) = f(a) = a^2 = 2$ so that $f\big(f(\sqrt{2})\big) = f(2) = \pm\sqrt{6}$, but then taking $x = \sqrt{2}$ in (2) yields a contradiction since $6 \neq \big(3\sqrt{2}-2\big)^2$. Similarly, the assumption $a = -\sqrt{2}$ leads to a contradiction. It follows that $a = 0$, hence $f(x) = x$ or $f(x) = -x$ for any given x. However, if $f(x_0) = x_0$ for some nonzero real number x_0, then $f\big(f(x_0)\big) = f(x_0) = x_0$, while from (1) we have $f\big(f(x_0)\big) = f\big(f(x_0-0)\big) = -f(x_0) = -x_0$. This is impossible since $x_0 \neq 0$, hence $f(x) = -x$ for all real x.

5. Find all triples of non-negative integers (x, y, n) such that $\dfrac{x!+y!}{n!} = 3^n$ (with the convention $0! = 1$).

Solved by Daniel Tsai, student, Taipei American School, Taipei, Taiwan; and Konstantine Zelator, University of Toledo, Toledo, OH, USA. We give the solution of Tsai.

Let S be the set of all ordered triples (x, y, n) of nonnegative integers such that $\dfrac{x!+y!}{n!} = 3^n$, or equivalently $x! + y! = 3^n n!$. For $n = 0$, it is seen at once that there is no corresponding $(x, y, n) \in S$. For $n = 1$, let $(x, y, n) \in S$; then if $x \geq 3$ or $y \geq 3$ we have $x! + y! \geq 7 > 3 = 3^1 1!$, thus x, $y < 3$ and simple checking yields that $\{(0,2,1),\ (1,2,1),\ (2,0,1),\ (2,1,1)\} \subset S$.

Lemma Let $(x, y, n) \in S$ with $n \geq 2$. Then x, $y \geq n$ and $x > n$ or $y > n$.

Proof: If x, $y \leq n$, then $x! + y! \leq 2n! < 3^n n!$, so $x > n$ or $y > n$. If $x > n$ and $y < n$, then $\dfrac{x!+y!}{n!} = \dfrac{x!}{n!} + \dfrac{y!}{n!} = 3^n$, a contradiction since $\dfrac{x!}{n!}$ is an integer but $\frac{y!}{n!}$ is not. Thus, if $x > n$, then $y \geq n$. Similarly, in the case $y > n$ we have $x \geq n$ by symmetry. ∎

We shall prove that for $n \geq 2$ there is no corresponding $(x, y, n) \in S$ by considering cases on n modulo 3.

Case 1. $n \equiv 0 \pmod 3$. Let $(x, y, n) \in S$ and assume without loss of generality that $x \leq y$. By the Lemma, $n \leq x \leq y$ and one of these two inequalities is strict. If $x > n$, then from $\frac{x! + y!}{n!} = 3^n$ it follows that $(n+1)|3^n$. However, $n+1$ has a prime divisor other than 3, a contradiction. Therefore, $n = x < y$, and consequently

$$\frac{x! + y!}{n!} = \frac{x!}{n!} + \frac{y!}{n!} = 1 + (n+1)(n+2)\cdots y = 3^n .$$

Thus, 3 divides $1+(n+1)(n+2)\cdots y$ and $(n+1)(n+2)\cdots y \equiv 2 \pmod 3$, which implies that $y = n+2$. However, $1 + (n+1)(n+2) < 3^n$ for $n \geq 3$ (by induction) and $1 + (2+1)(2+2) \neq 3^2$, contradicting the fact that $(x, y, n) \in S$.

Case 2. $n \equiv 1 \pmod 3$. Let $(x, y, n) \in S$ and assume without loss of generality that $x \leq y$. By reasoning similar to that of Case 1 it follows that $x = n$ and $y = n+1$. However, $1 + (n+1) < 3^n$ for each integer $n \geq 2$, contradicting the fact that $(x, y, n) \in S$.

Case 3. $n \equiv 2 \pmod 3$. Let $(x, y, n) \in S$ and assume without loss of generality that $x \leq y$. By the Lemma, $n \leq x \leq y$ and one of these two inequalities is strict. If $x \geq n+2$, then from $\frac{x! + y!}{n!} = 3^n$ it follows that $(n+2)|3^n$. However, $n+2$ has a prime divisor other than 3, a contradiction, hence $n \leq x < n+2$.

If $x = n$, then $n = x < y$ and

$$\frac{x! + y!}{n!} = \frac{x!}{n!} + \frac{y!}{n!} = 1 + (n+1)(n+2)\cdots y = 3^n ,$$

contradicting $n+1 \equiv 0 \pmod 3$.

If $x = n+1$, then

$$\frac{x! + y!}{n!} = \frac{x!}{n!} + \frac{y!}{n!} = (n+1) + (n+1)(n+2)\cdots y = 3^n .$$

If furthermore $y = n+1$, then

$$2(n+1) = (n+1) + (n+1)(n+2)\cdots y = 3^n$$

is even, a contradiction. Thus, $y > n+1$ and $(n+1)\big(1+(n+2)\ldots y\big) = 3^n$. It follows that $1+(n+2)\cdots y \equiv 0 \pmod 3$ and $(n+2)\cdots y \equiv 2 \pmod 3$, which implies that $y = n+3$. However, $(n+1)\big(1+(n+2)(n+3)\big) \neq 3^n$

for $1 \le n \le 5$ and by induction $(n+1)\bigl(1+(n+2)(n+3)\bigr) < 3^n$ for $n \ge 6$, contradicting the fact that $(x,y,n) \in S$.

6. Let the sequence x_1, x_2, x_3, $\dots$, be defined by $x_1 = a$, where a is a real number, and the recursion $x_{n+1} = 3x_n^3 - 7x_n^2 + 5x_n$ for $n \ge 1$.

Find all values of a for which the sequence has a finite limit as n tends to infinity, and find this limit.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Daniel Tsai, student, Taipei American School, Taipei, Taiwan. We give Bataille's write-up.

Let $f(x) = 3x^3 - 7x^2 + 5x$ and $g(x) = f(x) - x = x(x-1)(3x-4)$. The sequence $\{x_n\}$, which satisfies $x_{n+1} = f(x_n)$ for all positive integers n, can only converge to a root of $g(x) = 0$. Thus, the only possible finite limits of $\{x_n\}$ are 0, 1, and $\frac{4}{3}$. We show that the sequence is convergent if and only if $0 \le a \le \frac{4}{3}$, in which case the limit is 1 except if $a = 0$ and $\lim\limits_{n\to\infty} x_n = 0$ or if $a = \frac{4}{3}$ and $\lim\limits_{n\to\infty} x_n = \frac{4}{3}$.

Suppose first $a < 0$. Since $g(x) < 0$ when $x < 0$, it follows that $x_n < x_1 = a < 0$ for all positive integers n. If $\{x_n\}$ had a finite limit, ℓ, we would have $\ell \le a$, contradicting the fact that $\ell \in \{0,\,1,\,\frac{4}{3}\}$. Thus, $\{x_n\}$ is divergent when $a < 0$. Using the fact that $g(x) > 0$ for $x > \frac{4}{3}$, similar reasoning shows that $\{x_n\}$ is divergent when $a > \frac{4}{3}$.

If $a \in \{0,\,1,\,\frac{4}{3}\}$, then the sequence $\{x_n\}$ is constant.

If $a \in (1, \frac{4}{3})$, then using $f(x) - 1 = (x-1)^2(3x-1)$ an easy induction shows that $1 < x_{n+1} < x_n$ for all positive integers n. Thus, $\{x_n\}$ is decreasing and bounded, hence convergent. Its limit ℓ satisfies $\ell \ge 1$ and $\ell \in \{0,\,1,\,\frac{4}{3}\}$, that is, $\ell = 1$.

If $a \in [\frac{1}{3}, 1)$ then $x_2 = f(a) \ge 1$ and $x_2 < \frac{4}{3}$, as the maximum of f on $[0,1]$ is $f\bigl(\frac{5}{9}\bigr) = \frac{275}{243} < \frac{4}{3}$. From the previous case, we see that $\lim\limits_{n\to\infty} x_n = 1$.

It remains to study the case $a \in \bigl(0, \frac{1}{3}\bigr)$. Then, $\dfrac{1}{3^{m+1}} \le a < \dfrac{1}{3^m}$ for some unique positive integer m. If any of the numbers x_2, x_3, $\dots$, x_m is not less than $\frac{1}{3}$, let x_k be the one with the smallest index. Then $\frac{1}{3} \le x_k < \frac{4}{3}$ and by the previous cases $\{x_n\}_{n\ge k}$ converges to 1 and $\lim\limits_{n\to\infty} x_n = 1$. Otherwise, noting that $f(x) - 3x = x(x-2)(3x-1)$ is positive for $x \in \bigl(0, \frac{1}{3}\bigr)$, we have

$$\begin{aligned} x_2 &= f(x_1) > 3x_1 = 3a \ge \frac{1}{3^m}\,, \\ x_3 &= f(x_2) > 3x_2 \ge \frac{1}{3^{m-1}}\,, \\ &\cdots \\ x_m &= f(x_{m-1}) > 3x_{m-1} \ge \frac{1}{3^2}\,, \end{aligned}$$

and finally $x_{m+1} > \frac{1}{3}$. So $\{x_n\}_{n\ge m+1}$ converges to 1 and again $\lim\limits_{n\to\infty} x_n = 1$.

To finish this number of the *Corner* we give solutions from the readers to problems of the 2005 German Mathematical Olympiad, given at [2008 : 82].

1. Determine all pairs (x, y) of reals, which satisfy the system of equations

$$\begin{aligned} x^3 + 1 - xy^2 - y^2 &= 0\,, \\ y^3 + 1 - x^2 y - x^2 &= 0\,. \end{aligned}$$

Solved by George Apostolopoulos, Messolonghi, Greece; Konstantine Zelator, University of Toledo, Toledo, OH, USA; and Titu Zvonaru, Cománeşti, Romania. We give the write-up of Apostolopoulos.

We subtract the two equations of the system to obtain

$$\left(x^3 - y^3\right) \;+\; xy(x-y) \;+\; x^2 - y^2 \;=\; 0\,,$$

which upon factoring becomes

$$(x-y)(x+y)(x+y+1) \;=\; 0\,.$$

Thus, $y = x$ or $y = -x$ or $y = -x - 1$.

If $y = x$, then $x = \pm 1$, hence $(x,y) = (1,1)$ or $(x,y) = (-1,-1)$.
If $y = -x$, then again $x = \pm 1$, hence $(x,y) = (1,-1)$ or $(x,y) = (-1,1)$.
If $y = -x - 1$ we substitute into the first equation to obtain

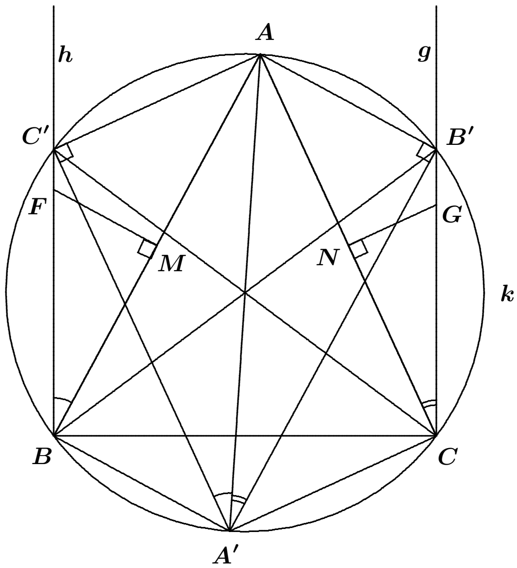
$$x^3 + 1 - x(x+1)^2 - (x+1)^2 \;=\; -3x(x+1) \;=\; 0\,,$$

hence $x = 0$ or $x = -1$ and $(x,y) = (0,-1)$ or $(x,y) = (-1,0)$.

2. Let A, B, and C be three distinct points on the circle k. Let the lines h and g each be perpendicular to BC with h passing through B and g passing through C. The perpendicular bisector of AB meets h in F and the perpendicular bisector of AC meets g in G. Prove that the product $|BF| \cdot |CG|$ is independent of the choice of A, whenever B and C are fixed.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Konstantine Zelator, University of Toledo, Toledo, OH, USA; and Titu Zvonaru, Cománeşti, Romania. We give the solution of Amengual Covas.

Let A' be the point diametrically opposite to A. Let M and N be the midpoints of the segments AB and AC, respectively. Let B' and C' be the second points of intersection of the lines g and h with the circle k, respectively.



Since $C'BC$ and BCB' are right angles, the segments BB' and CC' are diameters of the circle k. Thus, the two quadrilaterals $AC'A'C$ and $ABA'B'$ are rectangles, and hence parallelograms, so we have

$$|C'A'| = |CA| \quad \text{and} \quad |A'B'| = |AB|\,.$$

Since the right triangles BMF and $A'C'A$ are similar, as are the right triangles CNG and $A'B'A$, we have

$$\frac{BF}{AA'} = \frac{BM}{C'A'} = \frac{\frac{1}{2}AB}{CA}$$

and

$$\frac{CG}{AA'} = \frac{CN}{A'B'} = \frac{\frac{1}{2}CA}{AB}\,,$$

from which we obtain

$$BF \cdot CG = \frac{1}{4}|AA'|^2$$

as the square of the radius of k, which is independent of the choice of A.

3. A lamp is placed at each lattice point (x, y) in the plane (that is, x and y are both integers). At time $t = 0$ exactly one lamp is switched on. At any integer time $t \geq 1$, exactly those lamps are switched on which are at a distance of 2005 from some lamp which is already switched on. Prove that every lamp will be switched on at some time.

Solution by Titu Zvonaru, Comănești, Romania.

Assume that at time $t = 0$ the lamp at $O(0, 0)$ is switched on. Since $2005 = \sqrt{1357^2 + 1476^2}$ then at some time the lamps at the following lattice points will be switched on:

$$\begin{array}{rl} A_1(1357, 1476)\,, & O_1(2\cdot 1357, 0)\,, \\ A_2(3\cdot 1357, 1476)\,, & O_2(4\cdot 1357, 0)\,, \\ \vdots & \vdots \\ A_k\big((2k-1)\cdot 1357, 1476\big)\,, & O_k(2k\cdot 1357, 0)\,; \end{array}$$

and then the lamps at these lattice points will be switched on:

$$\begin{array}{l} B_1(2k\cdot 1357 - 2005, 0)\,, \\ B_2(2k\cdot 1357 - 2\cdot 2005, 0)\,, \\ B_3(2k\cdot 1357 - 3\cdot 2005, 0)\,, \\ \vdots \\ B_t(2k\cdot 1357 - t\cdot 2005, 0)\,. \end{array}$$

The equation $2k\cdot 1357 - 2005t = 1$ is the same as $2714k - 2005t = 1$, which has a solution in positive integers k, t because $\gcd(2714, 2005) = 1$, for example, $2714\cdot 1134 - 2005\cdot 1535 = 1$. Thus the lamp at $(1,0)$ will be switched on at some time. It follows (by symmetry) that every lamp will be switched on at some time.

4. Let $Q(n)$ denote the sum of the digits of the positive integer n. Prove that $Q\Big(Q\big(Q(2005^{2005})\big)\Big) = 7$.

Solution by Titu Zvonaru, Comănești, Romania.

It is well known that $Q(n) \equiv n \pmod 9$. Let $k = Q\Big(Q\big(Q(2005^{2005})\big)\Big)$, then $k \equiv 2005^{2005} \pmod 9$ and we have

$$\begin{array}{rcl} 2005^{2005} & \equiv & (-2)^{2005} \;=\; -2\cdot 2^{2004} \;=\; -2\cdot \left(2^3\right)^{668} \\ & \equiv & -2(-1)^{668} \;=\; -2 \;\equiv\; 7 \pmod 9\,, \end{array}$$

so that $k \equiv 7 \pmod 9$.

The number 2005^{2005} has at most $4\cdot 2005 = 8020$ digits. Hence, $Q(2005^{2005})$ is at most $9\cdot 8020 = 72180$. This implies that

$$Q\Big(Q\left(2005^{2005}\right)\Big) \;\le\; 5\cdot 9 \;=\; 45$$

and hence $k = Q\Big(Q\big(Q(2005^{2005})\big)\Big)$ is at most $Q(39) = 12$.

Altogether, k is an integer satisfying $0 \le k \le 12$ and $k \equiv 7 \pmod 9$, hence $k = 7$, as desired.

That completes the *Corner* for this issue. Send me your nice solutions, generalizations, and Olympiad problem sets.