SKOLIAD No. 114

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Please send your solutions to problems in this Skoliad by ***August 1, 2009***. Solutions should be sent to Lily Yen and Mogens Hansen at the address inside the back cover. The Skoliad section is in transition and, unfortunately, we have lost several of the submitted solutions to past contests. If you have copies of solutions that you sent to past contests, please send them again so that we can mention any correct solutions we receive. (This includes any contest in Skoliad appearing in or after the March 2008 issue of ***CRUX***).

Our first problem set of the year is the Math Kangaroo Contest Practice Set. The Kangaroo Contest is international in scope and supported in Canada by the Canadian Mathematical Society and the Institute of Electrical and Electronics Engineers (Northern Section).

Our thanks go to Valeria Pandelieva, the Canadian representative of the Kangaroo Contest, for bringing this contest to our attention, and for making us aware of the need for contests and math-participation in the lower years in Canada. For that reason, and also since this contest is straightforward to administer (see `www.mathkangaroocanada.com`), we are featuring its entire range of questions over all grades.

Finally, while it is a multiple choice test, we ask our readers to send in complete solutions showing all the steps and details so that we can evaluate the solutions and give full credit to the solvers.

Math Kangaroo Contest
Practice Set

Part A (3 points per question)

1. (Grades 3-4) In the addition example, each letter represents a digit. Equal digits are represented by the same letter. Different digits are represented by different letters. Which digit does the letter K represent?

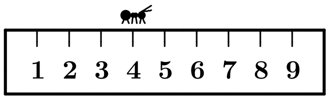
	O	K
$+$	K	O
W	O	W

(A) 0 (B) 1 (C) 2 (D) 8 (E) 9

2. (Grades 5-6) Ten caterpillars, arranged in a row one behind another, walked in the park. The length of each caterpillar was equal to 8 cm, and the distance any two adjacent caterpillars kept for safety reasons was 2 cm. What is the total length of their row?

(A) 100 cm (B) 98 cm (C) 82 cm (D) 102 cm (E) 96 cm

3. (Grades 7-8) An ant is running along a ruler of length 10 cm with a constant speed of 1 cm per second (see the figure). Any time when the ant reaches one of the ends of the ruler, it turns back and runs in the opposite direction. It takes the ant exactly 1 second to make a turn. The ant starts from the left end of the ruler. Nearest which number will it be after 2009 seconds?



(A) 1 cm (B) 2 cm (C) 3 cm (D) 4 cm (E) 5 cm

4. (Grades 9-10) Which of the numbers 2^6, 3^5, 4^4, 5^3, 6^2 is the greatest?

(A) 2^6 (B) 3^5 (C) 4^4 (D) 5^3 (E) 6^2

5. (Grades 11-12) A decorator has prepared a mixed paint, in which the volumes of red and yellow colours were in the ratio $2 : 3$. The resulting colour seemed too light to him, so he added 2 L of red paint. This way, the ratio of the volumes of the red and yellow colours changed to $3 : 2$. How many litres of paint did the decorator use?

(A) 5 L (B) 6 L (C) 7 L (D) 8 L (E) 9 L

Part B (4 points per question)

6. (Grades 3-4) Two boys are playing tennis until one of them wins four times. A tennis match cannot end in a draw. What is the greatest number of games they can play?

(A) 8 (B) 7 (C) 6 (D) 5 (E) 9

7. (Grades 5-6) In two years, my son will be twice as old as he was two years ago. In three years, my daughter will be three times as old as she was three years ago. Which of the following best describes the ages of the daughter and the son?

(A) The son is older; (B) The daughter is older; (C) They are twins;
(D) The son is twice as old as the daughter;
(E) The daughter is twice as old as the son.

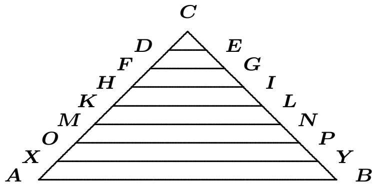
8. (Grades 7-8) Some points are marked on a straight line so that all distances 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm, 7 cm, and 9 cm are among the distances between these points. At least how many points are marked on the line?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

9. (Grades 9-10) Eva, Betty, Linda, and Cathy went to the cinema. Since it was not possible to buy four seats next to each other, they bought tickets for seats number 7 and 8 in the 10^{th} row and tickets for seats number 3 and 4 in the 12^{th} row. How many seating arrangements can they choose from, if Cathy does not want to sit next to Betty?

(A) 24 (B) 20 (C) 16 (D) 12 (E) 8

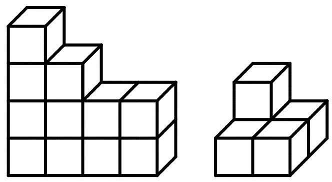
10. (Grades 11-12) Triangle ABC is isosceles with $BC = AC$. The segments DE, FG, HI, KL, MN, OP, and XY divide the sides AC and CB into equal parts. Find XY, if $AB = 40$ cm.



(A) 38 cm (B) 35 cm
(C) 33 cm (D) 30 cm (E) 27 cm

Part C (5 points per question)

11. (Grades 3-4) Matt and Nick constructed two buildings, shown in the figures, using identical cubes. Matt's building weighs 200 g, and Nick's building weighs 600 g. How many cubes from Nick's building are hidden and cannot be seen in the figure?



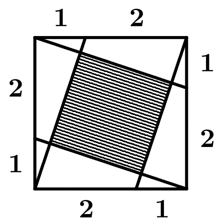
Nick's building Matt's building

(A) 1 (B) 2 (C) 3
(D) 4 (E) 5

12. (Grades 5-6) Consider all four-digit numbers divisible by 6 whose digits are in increasing order, from left to right. What is the hundreds digit of the largest such number?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

13. (Grades 7-8) A square of side length 3 is divided by several segments into polygons as shown in the figure. What percent of the area of the original square is the area of the shaded figure?

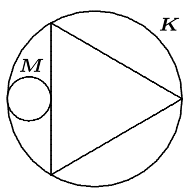


(A) 30% (B) $33\frac{1}{3}\%$ (C) 35%
(D) 40% (E) 50%

14. (Grades 9-10) A boy always tells the truth on Thursdays and Fridays, always tells lies on Tuesdays, and tells either truth or lies on the rest of the days of the week. Every day he was asked what his name was and six times in a row he gave the following answers: John, Bob, John, Bob, Pit, Bob. What did he answer on the seventh day?

(A) John (B) Bob (C) Pit (D) Kate
(E) Not enough information to decide

15. (Grades 11-12) An equilateral triangle and a circle M are inscribed in a circle K, as shown in the figure. What is the ratio of the area of K to the area of M?



(A) $8:1$ (B) $10:1$ (C) $12:1$
(D) $14:1$ (E) $16:1$

Concours Math Kangaroo
Feuille d'entraînement

Partie A (3 points par question)

1. (Classes 3-4) Dans l'exemple d'addition ci-dessus, chaque lettre différente représente un chiffre différent. Quel chiffre la lettre K représente-t-elle ?

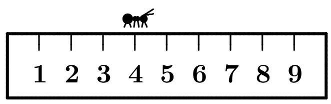
	O	K
$+$	K	O
W	O	W

(A) 0 (B) 1 (C) 2 (D) 8 (E) 9

2. (Classes 5-6) Dix chenilles se promenaient à la file indienne dans un parc. Chaque chenille mesurait 8 cm et, pour des raisons de sécurité, elles gardaient une distance de 2 cm entre chacune d'elles. Quelle était la longueur totale de leur cortège ?

(A) 100 cm (B) 98 cm (C) 82 cm (D) 102 cm (E) 96 cm

3. (Classes 7-8) Une fourmi court le long d'une règle de 10 cm de longueur, à la vitesse constante de 1 cm à la seconde (voir la figure). Chaque fois qu'elle atteint une extrémité, elle court dans la direction opposée et elle met exactement 1 seconde pour changer de direction. La fourmi part de l'extrémité gauche de la règle. Près de quel chiffre sera-telle après 2009 secondes ?



(A) 1 cm (B) 2 cm (C) 3 cm (D) 4 cm (E) 5 cm

4. (Classes 9-10) Lequel des nombres 2^6, 3^5, 4^4, 5^3, 6^2 est-il le plus grand ?

(A) 2^6 (B) 3^5 (C) 4^4 (D) 5^3 (E) 6^2

5. (Classes 11-12) Un décorateur a préparé un mélange de peinture où les volumes des couleurs rouge et jaune étaient dans un rapport de $2 : 3$. Trouvant le mélange trop clair, il ajouta 2 L de peinture rouge. Le rapport des volumes des couleurs rouge et jaune devint alors de $3 : 2$. Combien de litres de peinture le décorateur a-t-il utilisé ?

(A) 5 L (B) 6 L (C) 7 L (D) 8 L (E) 9 L

Partie B (4 points par question)

6. (Classes 3-4) Deux garçons jouent au tennis jusqu'à ce que l'un d'eux gagne quatre fois. Un match de tennis ne peut finir en un pointage nul. Quel est le plus grand nombre de jeux qu'ils peuvent jouer ?

(A) 8 (B) 7 (C) 6 (D) 5 (E) 9

7. (Classes 5-6) Dans deux ans, mon fils aura deux fois l'âge qu'il avait il y a deux ans. Dans trois ans, ma fille aura trois fois l'âge qu'elle avait il y a trois ans. Quelle réponse décrit-elle le mieux l'âge de la fille et du fils ?

(A) Le fils est plus âgé ; (B) La fille est plus âgée ;
(C) Ils sont des jumeaux ; (D) Le fils est deux fois plus âgé que la fille ;
(E) La fille est deux fois plus âgée que le fils.

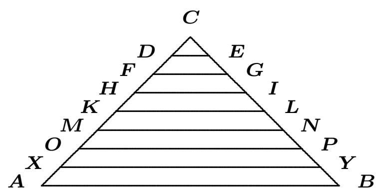
8. (Classes 7-8) Sur une droite on marque des points de sorte que toutes les distances de 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm, 7 cm et 9 cm figurent parmi les distances entre ces points. Combien y a-t-il au minimum de points marqués sur cette droite ?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

9. (Classes 9-10) Liliane, Nicole, Katia et Charlotte sont allées au cinéma. Comme il n'était pas possible d'acheter quatre places ensemble, elles ont acheté des billets pour les sièges numéro 7 et 8 dans la 10^e-rangée et d'autres pour les sièges numéro 3 et 4 dans la 12^e-rangée. De combien de manières peuvent-elles choisir de s'asseoir, si Charlotte ne veut pas être assise à côté de Nicole ?

(A) 24 (B) 20 (C) 16 (D) 12 (E) 8

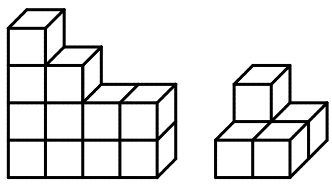
10. (Classes 11-12) Soit ABC un triangle isocèle avec $BC = AC$. Les segments DE, FG, HI, KL, MN, OP et XY divisent les côtés AC et CB en parties égales. Trouver XY si $AB = 40$ cm.



(A) 38 cm (B) 35 cm
(C) 33 cm (D) 30 cm (E) 27 cm

Partie C (5 points par question)

11. (Classes 3-4) En utilisant des cubes identiques, Mathieu et Nicolas ont construit deux bâtiments, comme illustrés dans les figures. Le bâtiment de Mathieu pèse 200 g et celui de Nicolas 600 g. Combien de cubes du bâtiment de Nicolas sont-ils cachés et ne peuvent être vus dans la figure ?



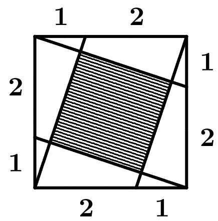
bâtiment de Nicolas bâtiment de Mathieu

(A) 1 (B) 2 (C) 3
(D) 4 (E) 5

12. (Classes 5-6) On considère tous les nombres de quatre chiffres, divisibles par 6 et dont les chiffres, lus de gauche à droite, vont en ordre croissant. Quel est le chiffre des centaines dans le plus grand de ces nombres ?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

13. (Classes 7-8) On divise un carré de côté 3 en polygones avec plusieurs segments comme indiqué dans la figure. Quel est le percentage de l'aire de la figure ombrée par rapport à celle du carré ?

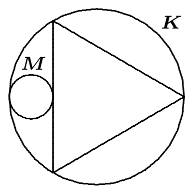


(A) 30% (B) $33\frac{1}{3}\%$ (C) 35%
(D) 40% (E) 50%

14. (Classes 9-10) Un garçon dit toujours la vérité les jeudis et vendredis, ment toujours les mardis et, les autres jours de la semaine, soit il dit la vérité soit il ment. On lui demanda son nom chaque jour de la semaine et les six premières fois, il donna les réponses suivantes : Jean, Bernard, Jean, Bernard, Paul, Bernard. Quelle fut sa réponse le septième jour ?

(A) Jean (B) Bernard (C) Paul (D) Luc
(E) Pas possible de décider

15. (Classes 11-12) On inscrit un triangle équilatéral et un cercle M dans un cercle K, comme indiqué dans la figure. Quel est le rapport de l'aire de K à celle de M ?



(A) $8 : 1$ (B) $10 : 1$ (C) $12 : 1$
(D) $14 : 1$ (E) $16 : 1$

Next we shall give solutions to the Mathematics Association of Quebec Contest (Secondary level) February 9, 2006 [2008 : 67-68]. We apologize to any readers who sent in solutions to this contest but whose solutions we have lost.

1. **A particular magic square**. It is well known that a magic square is obtained by putting numbers in a square such that the sum of each row, column, and diagonal is the same, as for example,

8	1	6
3	5	7
4	9	2

Imagine now that we decide to invent a new form of such squares by replacing the sum by a product. We ask you to find such a square by replacing the asterisks, $*$, by natural numbers, not necessarily distinct or consecutive, in the following square:

$*$	1	$*$
4	$*$	$*$
$*$	$*$	2

Solution by the editor.

Suppose that the square A below is a magic square. Then the square B is a magic square for products. For example, by the Law of Exponents, the product along the first row of B is $x^a x^b x^c = x^{a+b+c}$ and the product along the first column of B is $x^a x^d x^g = x^{a+d+g}$ and these are the same because $a + b + c = a + d + g$. The same is true for the other rows, columns, and diagonals of B.

$$A = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline \end{array}\,, \qquad B = \begin{array}{|c|c|c|}\hline x^a & x^b & x^c \\ \hline x^d & x^e & x^f \\ \hline x^g & x^h & x^i \\ \hline \end{array}\,.$$

Now, if we subtract 1 from every entry of the first square given in the question and if we take $x = 2$, then the square B below is a solution to the problem.

$$A = \begin{array}{|c|c|c|}\hline 7 & 0 & 5 \\ \hline 2 & 4 & 6 \\ \hline 3 & 8 & 1 \\ \hline \end{array}\,, \qquad B = \begin{array}{|c|c|c|}\hline 2^7 & 2^0 & 2^5 \\ \hline 2^2 & 2^4 & 2^6 \\ \hline 2^3 & 2^8 & 2^1 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 128 & 1 & 32 \\ \hline 4 & 16 & 64 \\ \hline 8 & 256 & 2 \\ \hline \end{array}\,.$$

2. **Clovis' outing**. Clovis likes to take an outing in the natural numbers. Each day, he starts with a natural number of his choice, the biggest possible. Then, during his day, he passes from number to number using the following rules. Suppose that the sequence of numbers is currently at n.

(1) If n is divisible by 3 without remainder, then the next number is $n/3$.

(2) If the remainder after dividing n by 3 is 1, then the next number is $2n + 1$.

(3) If the remainder after dividing n by 3 is 2, then the next number is $2n - 1$.

(4) If $n = 1$, then the sequence stops.

Over the years that he has played this game, he noticed that, whatever the starting number, the sequence always ended up with the number 1. However, he wonders if there is a sequence that increases indefinitely, with larger and larger numbers on average, or such that it ends up in a loop of numbers that does not contain 1. Determine if such a sequence is possible and give an example, or show that such a sequence does not exist by showing that all sequences using the above rules inevitably end up at the number 1.

Here is an example of such a sequence: Starting with 55, we get 111, 37, 75, 25, 51, 17, 33, 11, 21, 7, 15, 5, 9, 3 and 1, which ends the sequence.

Solution by the editor.

Note that Clovis' sequence starting with 55 has a decreasing subsequence that goes to 1, given by the underlined numbers: $\underline{55}$, 111, $\underline{37}$, 75, $\underline{25}$,

51, $\underline{17}$, 33, $\underline{11}$, 21, $\underline{7}$, 15, $\underline{5}$, 9, $\underline{3}$, $\underline{1}$. We will show that for any number $a > 1$ in one of Clovis' sequences, there is always a number b coming after a in the sequence such that $a > b$. Thus, if Clovis starts with $n > 1$, then there will be a subsequence n, m, p, $\ldots$ with $n > m > p > \cdots$ and this subsequence must eventually hit the number 1 (because all of the terms in it are positive, it cannot decrease forever).

If $a > 1$ and $a = 3k$, $k > 1$, then by rule (1) the number $b = k$ comes right after a and $a > b$.

If $a > 1$ and $a = 3k + 1$, $k > 0$, then by rule (2) the number $2a + 1 = 2(3k+1)+1 = 6k+3$ comes right after the number a, and then by rule (1) the number $(6k+3)/3 = 2k+1$ comes after $2a+1$. Since $a = 3k+1 > 2k+1$, we see that the number $b = 2k + 1$ comes after a and $a > b$.

If $a > 1$ and $a = 3k + 2$, $k \geq 0$, then by rule (3) the number $2a - 1 = 2(3k+2)+1 = 6k+3$ comes right after the number a, and then by rule (1) the number $(6k+3)/3 = 2k+1$ comes after $2a-1$. Since $a = 3k+2 > 2k+1$, we see that the number $b = 2k + 1$ comes after a and $a > b$.

Thus, in all cases where $a > 1$, there is a number b coming after a in the sequence such that $a > b$, and we are done.

3. **Eight balls in two urns**. We give you two similar urns, four white balls, and four black balls. You must separate the balls amongst the two urns (not necessarily the same number in each urn), after which both urns will be made indistinguishable. How should the balls be distributed to maximize the chances that, if you draw a ball randomly from a randomly chosen urn, you will obtain a white ball?

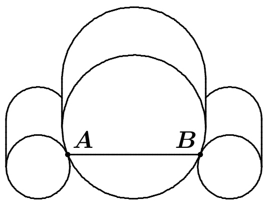
Solution by the editor.

Put 1 white ball in one urn and all the other balls in the other urn. The probability of choosing the urn with 1 white ball and then drawing that white ball from it is $\frac{1}{2}\cdot 1 = \frac{1}{2}$ and the probability of choosing the other urn and then drawing a white ball from it is $\frac{1}{2}\cdot\frac{3}{3+4} = \frac{3}{14}$. Thus, with this distribution, the overall probability of ultimately obtaining a white ball is $p = \frac{1}{2}\cdot 1+\frac{1}{2}\cdot\frac{3}{7} = \frac{5}{7}$.

Now let p_1 and p_2 be the probabilities of drawing white balls from the two urns ($p_1 = 1$ and $p_2 = \frac{3}{7}$ above). The overall probability of ultimately obtaining a white ball is then $p = \frac{1}{2}p_1 + \frac{1}{2}p_2 = \dfrac{p_1 + p_2}{2}$, which is the average of the probabilities p_1 and p_2. Therefore, $p > \frac{5}{7}$ implies that $p_1 > \frac{5}{7}$ or $p_2 > \frac{5}{7}$. To make an urn with $p_1 > \frac{5}{7}$ (say) we must have $(w, b) = (1, 0)$, $(2, 0)$, $(3, 0)$, $(3, 1)$, $(4, 0)$, or $(4, 1)$, where the urn contains w white balls and b black balls. These are the only distributions that could yield $p > \frac{5}{7}$. In the case of $(w, b) = (2, 0)$, $(3, 0)$, or $(4, 0)$ we have $p < \frac{5}{7}$, as moving all but one white ball to the other urn increases p_2 but leaves $p_1 = 1$. Finally, $(w, b) = (3, 1)$ or $(4, 1)$ yields $p = \frac{1}{2}$ or $p = \frac{2}{5}$, each less than $\frac{5}{7}$.

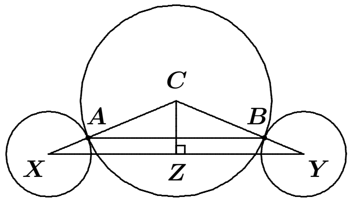
Our first distribution maximizes our chance of obtaining a white ball.

4. The three attached barrels. Three big cylindrical barrels, lying parallel to the earth, are attached by a steel cable at their contact points, A and B, such that they stay fixed in place. Knowing that the two smaller ones each have a radius of 4 metres and the biggest one has a radius of 9 metres, what is the length of the steel cable?

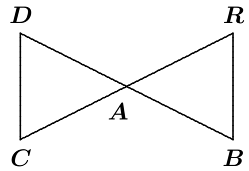


Solution by the editor.

Join the centres of the smaller barrels and drop a perpendicular to this segment from the centre of the larger barrel, as in the diagram at right. Since the barrels rest on the earth, the length of CZ is the difference of their radii, that is, $|CZ| = 9 - 4 = 5$. Also the length of CY is the sum of the radii, that is, $|CY| = 9 + 4 = 13$. By the Pythagorean Theorem $|ZY| = \sqrt{13^2 - 5^2} = 12$. Thus, $|XY| = 24$. Finally, since triangle CAB is similar to triangle CXY, we have $|AB| = \frac{9}{13}|XY| = \frac{216}{13}$.



5. The magic words. An illusionist is searching for magic words to accompany his many magic tricks. He decides to construct his magic words starting with the diagram on the right. He takes a path through the diagram and jots downs the letters he finds on it. Each magic word must have exactly 11 letters and must start and end with the letter A. Two consecutive letters must never be identical. How many magic words are there?



Note: Here are two possible magic words: $ABRACADABRA$ and $ARADCABARBA$.

Solution by the editor.

Let M_k be the number of k-letter words that start with A and end with A and that can be formed by travelling through the bowtie. Let N_k be the number of k-letter words starting with A but *not* ending with the letter A that can be similarly formed.

If $k \geq 2$, then we see that $M_k = N_{k-1}$, because removing the letter A from a word ending with A leaves a word not ending in A (but still starting with A) and the process can be reversed. Similarly, by deleting the last letter of a word of length k that starts with A but does not end in A, we see that $N_k = N_{k-1} + 4M_{k-1}$, because any of the four letters different from A can be added to a word not ending in A or else there is only way to extend a $(k-1)$-letter word not ending in A to one that still does not end in A. Since $M_{k-1} = N_{k-2}$ the last equation becomes $N_k = N_{k-1} + 4N_{k-2}$, where $k \geq 1$.

We now have $N_1 = 0$, $N_2 = 4$, $N_3 = N_2 + 4N_1 = 4 + 4 \cdot 0 = 4$, and so forth. The results of calculating the N_i are summarized in the table below:

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}
0	4	4	20	36	116	260	724	1764	4660

Finally $M_{11} = N_{10}$, so there are 4660 magic words altogether.

6. **All ten digits**. Find the smallest positive natural number N such that, in the decimal notation, N and $2N$ together use all ten digits: 0, 1, 2, $\ldots$, 9.

Solution by the editor.

We have $2(13485) = 26970$, and we will prove that if N_1 and $2N_1$ together use all ten digits and $N_1 \leq N = 13485$, then $N_1 = N$.

As N_1 has five digits and $N_1 \leq N$, then $N_1 = 1\ldots$ and $2N_1 = 2\ldots$. Digits 1 and 2 are now used and $2N_1$ uses 0 (otherwise N_1 uses 0 and $2N_1$ then uses 0 or 1, a contradiction). Thus, $N_1 = 13\ldots$. The smallest available digit for N_1 is now a 4 and $N_1 \leq N$, hence $N_1 = 134\ldots$ and $2N_1 = 26\ldots$. The number N_1 uses 5, because $2N_1$ uses 0. If $N_1 = 1345x$, then x is a digit greater than 5 and $2N_1 = 2691y$, a contradiction. Thus, $N_1 = 134x5 \leq N$. Finally, $x \neq 7$, hence $x = 8$.

Therefore, $N_1 = N$ and N is the smallest positive integer with the given property.

[*Ed.:* Rolland Gaudet offers the solution $N = 6792$ if initial zeroes are allowed, for then $2(6792) = 013584$.]

7. **The pizza toppings**. At the Julio pizzeria, all the pizzas have cheese and tomato sauce on them. The choice of toppings is limited to black olives, anchovies, and sausage. Of the 200 clients Julio had yesterday, 40 took anchovies, 80 took black olives, 120 took sausage, 60 took at the same time black olives and sausage, but none took at the same time anchovies and black olives or anchovies and sausage. How many clients took none of the three toppings?

Solution by the editor.

Let t be the number of customers who took at least one topping. Any customer who took anchovies took no other topping, so $t = 40 + x$ where x is the number of customers who took black olives or sausage (or both). There were 60 customers who took both black olives and sausage, so $20 = 80 - 60$ took just black olives and nothing else. Similarly, $60 = 120 - 60$ customers took sausage and nothing else. Thus, $t = 40 + x = 40 + (20 + 60 + 60) = 180$, and the number of customers who took no toppings is $200 - t = 20$.