SKOLIAD No. 114

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Please send your solutions to problems in this Skoliad by ***August 1, 2009***. Solutions should be sent to Lily Yen and Mogens Hansen at the address inside the back cover. The Skoliad section is in transition and, unfortunately, we have lost several of the submitted solutions to past contests. If you have copies of solutions that you sent to past contests, please send them again so that we can mention any correct solutions we receive. (This includes any contest in Skoliad appearing in or after the March 2008 issue of ***CRUX***).

Our first problem set of the year is the Math Kangaroo Contest Practice Set. The Kangaroo Contest is international in scope and supported in Canada by the Canadian Mathematical Society and the Institute of Electrical and Electronics Engineers (Northern Section).

Our thanks go to Valeria Pandelieva, the Canadian representative of the Kangaroo Contest, for bringing this contest to our attention, and for making us aware of the need for contests and math-participation in the lower years in Canada. For that reason, and also since this contest is straightforward to administer (see www.mathkangaroocanada.com), we are featuring its entire range of questions over all grades.

Finally, while it is a multiple choice test, we ask our readers to send in complete solutions showing all the steps and details so that we can evaluate the solutions and give full credit to the solvers.

Math Kangaroo Contest
Practice Set

Part A (3 points per question)

1. (Grades 3-4) In the addition example, each letter represents a digit. Equal digits are represented by the same letter. Different digits are represented by different letters. Which digit does the letter K represent?

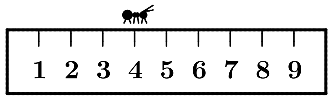
	O	K
$+$	K	O
W	O	W

(A) 0 (B) 1 (C) 2 (D) 8 (E) 9

2. (Grades 5-6) Ten caterpillars, arranged in a row one behind another, walked in the park. The length of each caterpillar was equal to 8 cm, and the distance any two adjacent caterpillars kept for safety reasons was 2 cm. What is the total length of their row?

(A) 100 cm (B) 98 cm (C) 82 cm (D) 102 cm (E) 96 cm

3. (Grades 7-8) An ant is running along a ruler of length 10 cm with a constant speed of 1 cm per second (see the figure). Any time when the ant reaches one of the ends of the ruler, it turns back and runs in the opposite direction. It takes the ant exactly 1 second to make a turn. The ant starts from the left end of the ruler. Nearest which number will it be after 2009 seconds?



(A) 1 cm (B) 2 cm (C) 3 cm (D) 4 cm (E) 5 cm

4. (Grades 9-10) Which of the numbers 2^6, 3^5, 4^4, 5^3, 6^2 is the greatest?

(A) 2^6 (B) 3^5 (C) 4^4 (D) 5^3 (E) 6^2

5. (Grades 11-12) A decorator has prepared a mixed paint, in which the volumes of red and yellow colours were in the ratio $2 : 3$. The resulting colour seemed too light to him, so he added 2 L of red paint. This way, the ratio of the volumes of the red and yellow colours changed to $3 : 2$. How many litres of paint did the decorator use?

(A) 5 L (B) 6 L (C) 7 L (D) 8 L (E) 9 L

Part B (4 points per question)

6. (Grades 3-4) Two boys are playing tennis until one of them wins four times. A tennis match cannot end in a draw. What is the greatest number of games they can play?

(A) 8 (B) 7 (C) 6 (D) 5 (E) 9

7. (Grades 5-6) In two years, my son will be twice as old as he was two years ago. In three years, my daughter will be three times as old as she was three years ago. Which of the following best describes the ages of the daughter and the son?

(A) The son is older; (B) The daughter is older; (C) They are twins;
(D) The son is twice as old as the daughter;
(E) The daughter is twice as old as the son.

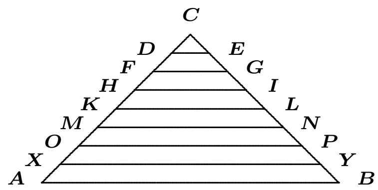
8. (Grades 7-8) Some points are marked on a straight line so that all distances 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm, 7 cm, and 9 cm are among the distances between these points. At least how many points are marked on the line?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

9. (Grades 9-10) Eva, Betty, Linda, and Cathy went to the cinema. Since it was not possible to buy four seats next to each other, they bought tickets for seats number 7 and 8 in the 10^{th} row and tickets for seats number 3 and 4 in the 12^{th} row. How many seating arrangements can they choose from, if Cathy does not want to sit next to Betty?

(A) 24 (B) 20 (C) 16 (D) 12 (E) 8

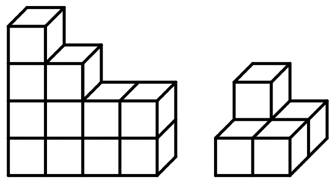
10. (Grades 11-12) Triangle ABC is isosceles with $BC = AC$. The segments DE, FG, HI, KL, MN, OP, and XY divide the sides AC and CB into equal parts. Find XY, if $AB = 40$ cm.



(A) 38 cm (B) 35 cm
(C) 33 cm (D) 30 cm (E) 27 cm

Part C (5 points per question)

11. (Grades 3-4) Matt and Nick constructed two buildings, shown in the figures, using identical cubes. Matt's building weighs 200 g, and Nick's building weighs 600 g. How many cubes from Nick's building are hidden and cannot be seen in the figure?



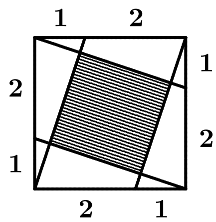
Nick's building Matt's building

(A) 1 (B) 2 (C) 3
(D) 4 (E) 5

12. (Grades 5-6) Consider all four-digit numbers divisible by 6 whose digits are in increasing order, from left to right. What is the hundreds digit of the largest such number?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

13. (Grades 7-8) A square of side length 3 is divided by several segments into polygons as shown in the figure. What percent of the area of the original square is the area of the shaded figure?

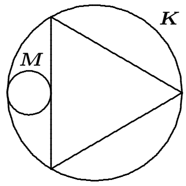


(A) 30% (B) $33\frac{1}{3}\%$ (C) 35%
(D) 40% (E) 50%

14. (Grades 9-10) A boy always tells the truth on Thursdays and Fridays, always tells lies on Tuesdays, and tells either truth or lies on the rest of the days of the week. Every day he was asked what his name was and six times in a row he gave the following answers: John, Bob, John, Bob, Pit, Bob. What did he answer on the seventh day?

(A) John (B) Bob (C) Pit (D) Kate
(E) Not enough information to decide

15. (Grades 11-12) An equilateral triangle and a circle M are inscribed in a circle K, as shown in the figure. What is the ratio of the area of K to the area of M?



(A) $8:1$ (B) $10:1$ (C) $12:1$
(D) $14:1$ (E) $16:1$

Concours Math Kangaroo
Feuille d'entraînement

Partie A (3 points par question)

1. (Classes 3-4) Dans l'exemple d'addition ci-dessus, chaque lettre différente représente un chiffre différent. Quel chiffre la lettre K représente-t-elle ?

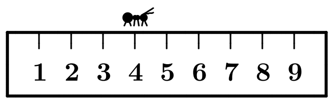
	O	*K*
+	*K*	*O*
W	*O*	*W*

(A) 0 (B) 1 (C) 2 (D) 8 (E) 9

2. (Classes 5-6) Dix chenilles se promenaient à la file indienne dans un parc. Chaque chenille mesurait 8 cm et, pour des raisons de sécurité, elles gardaient une distance de 2 cm entre chacune d'elles. Quelle était la longueur totale de leur cortège ?

(A) 100 cm (B) 98 cm (C) 82 cm (D) 102 cm (E) 96 cm

3. (Classes 7-8) Une fourmi court le long d'une règle de 10 cm de longueur, à la vitesse constante de 1 cm à la seconde (voir la figure). Chaque fois qu'elle atteint une extrémité, elle court dans la direction opposée et elle met exactement 1 seconde pour changer de direction. La fourmi part de l'extrémité gauche de la règle. Près de quel chiffre sera-telle après 2009 secondes ?



(A) 1 cm (B) 2 cm (C) 3 cm (D) 4 cm (E) 5 cm

4. (Classes 9-10) Lequel des nombres 2^6, 3^5, 4^4, 5^3, 6^2 est-il le plus grand ?

(A) 2^6 (B) 3^5 (C) 4^4 (D) 5^3 (E) 6^2

5. (Classes 11-12) Un décorateur a préparé un mélange de peinture où les volumes des couleurs rouge et jaune étaient dans un rapport de $2 : 3$. Trouvant le mélange trop clair, il ajouta 2 L de peinture rouge. Le rapport des volumes des couleurs rouge et jaune devint alors de $3 : 2$. Combien de litres de peinture le décorateur a-t-il utilisé ?

(A) 5 L (B) 6 L (C) 7 L (D) 8 L (E) 9 L

Partie B (4 points par question)

6. (Classes 3-4) Deux garçons jouent au tennis jusqu'à ce que l'un d'eux gagne quatre fois. Un match de tennis ne peut finir en un pointage nul. Quel est le plus grand nombre de jeux qu'ils peuvent jouer ?

(A) 8 (B) 7 (C) 6 (D) 5 (E) 9

7. (Classes 5-6) Dans deux ans, mon fils aura deux fois l'âge qu'il avait il y a deux ans. Dans trois ans, ma fille aura trois fois l'âge qu'elle avait il y a trois ans. Quelle réponse décrit-elle le mieux l'âge de la fille et du fils ?

(A) Le fils est plus âgé ; (B) La fille est plus âgée ;
(C) Ils sont des jumeaux ; (D) Le fils est deux fois plus âgé que la fille ;
(E) La fille est deux fois plus âgée que le fils.

8. (Classes 7-8) Sur une droite on marque des points de sorte que toutes les distances de 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm, 7 cm et 9 cm figurent parmi les distances entre ces points. Combien y a-t-il au minimum de points marqués sur cette droite ?

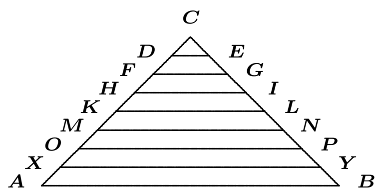
(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

9. (Classes 9-10) Liliane, Nicole, Katia et Charlotte sont allées au cinéma. Comme il n'était pas possible d'acheter quatre places ensemble, elles ont acheté des billets pour les sièges numéro 7 et 8 dans la 10^e-rangée et d'autres pour les sièges numéro 3 et 4 dans la 12^e-rangée. De combien de manières peuvent-elles choisir de s'asseoir, si Charlotte ne veut pas être assise à côté de Nicole ?

(A) 24 (B) 20 (C) 16 (D) 12 (E) 8

10. (Classes 11-12) Soit ABC un triangle isocèle avec $BC = AC$. Les segments DE, FG, HI, KL, MN, OP et XY divisent les côtés AC et CB en parties égales. Trouver XY si $AB = 40$ cm.

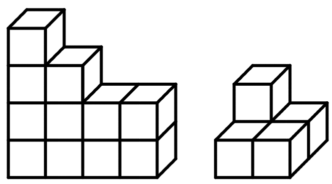
(A) 38 cm (B) 35 cm
(C) 33 cm (D) 30 cm (E) 27 cm



Partie C (5 points par question)

11. (Classes 3-4) En utilisant des cubes identiques, Mathieu et Nicolas ont construit deux bâtiments, comme illustrés dans les figures. Le bâtiment de Mathieu pèse 200 g et celui de Nicolas 600 g. Combien de cubes du bâtiment de Nicolas sont-ils cachés et ne peuvent être vus dans la figure ?

(A) 1 (B) 2 (C) 3
(D) 4 (E) 5

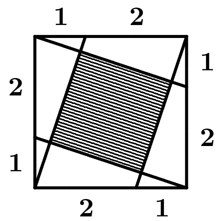


bâtiment de Nicolas — bâtiment de Mathieu

12. (Classes 5-6) On considère tous les nombres de quatre chiffres, divisibles par 6 et dont les chiffres, lus de gauche à droite, vont en ordre croissant. Quel est le chiffre des centaines dans le plus grand de ces nombres ?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

13. (Classes 7-8) On divise un carré de côté 3 en polygones avec plusieurs segments comme indiqué dans la figure. Quel est le percentage de l'aire de la figure ombrée par rapport à celle du carré ?

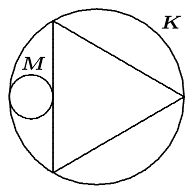


(A) 30% (B) $33\frac{1}{3}\%$ (C) 35%
(D) 40% (E) 50%

14. (Classes 9-10) Un garçon dit toujours la vérité les jeudis et vendredis, ment toujours les mardis et, les autres jours de la semaine, soit il dit la vérité soit il ment. On lui demanda son nom chaque jour de la semaine et les six premières fois, il donna les réponses suivantes : Jean, Bernard, Jean, Bernard, Paul, Bernard. Quelle fut sa réponse le septième jour ?

(A) Jean (B) Bernard (C) Paul (D) Luc
(E) Pas possible de décider

15. (Classes 11-12) On inscrit un triangle équilatéral et un cercle M dans un cercle K, comme indiqué dans la figure. Quel est le rapport de l'aire de K à celle de M ?



(A) $8:1$ (B) $10:1$ (C) $12:1$
(D) $14:1$ (E) $16:1$

Next we shall give solutions to the Mathematics Association of Quebec Contest (Secondary level) February 9, 2006 [2008 : 67-68]. We apologize to any readers who sent in solutions to this contest but whose solutions we have lost.

1. **A particular magic square**. It is well known that a magic square is obtained by putting numbers in a square such that the sum of each row, column, and diagonal is the same, as for example,

8	1	6
3	5	7
4	9	2

Imagine now that we decide to invent a new form of such squares by replacing the sum by a product. We ask you to find such a square by replacing the asterisks, $*$, by natural numbers, not necessarily distinct or consecutive, in the following square:

$*$	1	$*$
4	$*$	$*$
$*$	$*$	2

Solution by the editor.

Suppose that the square A below is a magic square. Then the square B is a magic square for products. For example, by the Law of Exponents, the product along the first row of B is $x^a x^b x^c = x^{a+b+c}$ and the product along the first column of B is $x^a x^d x^g = x^{a+d+g}$ and these are the same because $a + b + c = a + d + g$. The same is true for the other rows, columns, and diagonals of B.

$$A = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline \end{array}\,, \qquad B = \begin{array}{|c|c|c|}\hline x^a & x^b & x^c \\ \hline x^d & x^e & x^f \\ \hline x^g & x^h & x^i \\ \hline \end{array}\,.$$

Now, if we subtract 1 from every entry of the first square given in the question and if we take $x = 2$, then the square B below is a solution to the problem.

$$A = \begin{array}{|c|c|c|}\hline 7 & 0 & 5 \\ \hline 2 & 4 & 6 \\ \hline 3 & 8 & 1 \\ \hline \end{array}\,, \qquad B = \begin{array}{|c|c|c|}\hline 2^7 & 2^0 & 2^5 \\ \hline 2^2 & 2^4 & 2^6 \\ \hline 2^3 & 2^8 & 2^1 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 128 & 1 & 32 \\ \hline 4 & 16 & 64 \\ \hline 8 & 256 & 2 \\ \hline \end{array}\,.$$

2. **Clovis' outing**. Clovis likes to take an outing in the natural numbers. Each day, he starts with a natural number of his choice, the biggest possible. Then, during his day, he passes from number to number using the following rules. Suppose that the sequence of numbers is currently at n.

(1) If n is divisible by 3 without remainder, then the next number is $n/3$.

(2) If the remainder after dividing n by 3 is 1, then the next number is $2n + 1$.

(3) If the remainder after dividing n by 3 is 2, then the next number is $2n - 1$.

(4) If $n = 1$, then the sequence stops.

Over the years that he has played this game, he noticed that, whatever the starting number, the sequence always ended up with the number 1. However, he wonders if there is a sequence that increases indefinitely, with larger and larger numbers on average, or such that it ends up in a loop of numbers that does not contain 1. Determine if such a sequence is possible and give an example, or show that such a sequence does not exist by showing that all sequences using the above rules inevitably end up at the number 1.

Here is an example of such a sequence: Starting with 55, we get 111, 37, 75, 25, 51, 17, 33, 11, 21, 7, 15, 5, 9, 3 and 1, which ends the sequence.

Solution by the editor.

Note that Clovis' sequence starting with 55 has a decreasing subsequence that goes to 1, given by the underlined numbers: $\underline{55}$, 111, $\underline{37}$, 75, $\underline{25}$, 51, <u>17</u>, 33, <u>11</u>, 21, <u>7</u>, 15, <u>5</u>, 9, <u>3</u>, <u>1</u>. We will show that for any number $a > 1$ in one of Clovis' sequences, there is always a number b coming after a in the sequence such that $a > b$. Thus, if Clovis starts with $n > 1$, then there will be a subsequence n, m, p, ... with $n > m > p > \cdots$ and this subsequence must eventually hit the number 1 (because all of the terms in it are positive, it cannot decrease forever).

If $a > 1$ and $a = 3k$, $k > 1$, then by rule (1) the number $b = k$ comes right after a and $a > b$.

If $a > 1$ and $a = 3k + 1$, $k > 0$, then by rule (2) the number $2a + 1 = 2(3k+1)+1 = 6k+3$ comes right after the number a, and then by rule (1) the number $(6k+3)/3 = 2k+1$ comes after $2a+1$. Since $a = 3k+1 > 2k+1$, we see that the number $b = 2k + 1$ comes after a and $a > b$.

If $a > 1$ and $a = 3k + 2$, $k \geq 0$, then by rule (3) the number $2a - 1 = 2(3k+2)+1 = 6k+3$ comes right after the number a, and then by rule (1) the number $(6k+3)/3 = 2k+1$ comes after $2a-1$. Since $a = 3k+2 > 2k+1$, we see that the number $b = 2k + 1$ comes after a and $a > b$.

Thus, in all cases where $a > 1$, there is a number b coming after a in the sequence such that $a > b$, and we are done.

3. **Eight balls in two urns**. We give you two similar urns, four white balls, and four black balls. You must separate the balls amongst the two urns (not necessarily the same number in each urn), after which both urns will be made indistinguishable. How should the balls be distributed to maximize the chances that, if you draw a ball randomly from a randomly chosen urn, you will obtain a white ball?

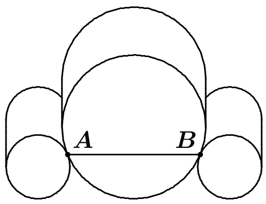
Solution by the editor.

Put 1 white ball in one urn and all the other balls in the other urn. The probability of choosing the urn with 1 white ball and then drawing that white ball from it is $\frac{1}{2} \cdot 1 = \frac{1}{2}$ and the probability of choosing the other urn and then drawing a white ball from it is $\frac{1}{2} \cdot \frac{3}{3+4} = \frac{3}{14}$. Thus, with this distribution, the overall probability of ultimately obtaining a white ball is $p = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{3}{7} = \frac{5}{7}$.

Now let p_1 and p_2 be the probabilities of drawing white balls from the two urns ($p_1 = 1$ and $p_2 = \frac{3}{7}$ above). The overall probability of ultimately obtaining a white ball is then $p = \frac{1}{2}p_1 + \frac{1}{2}p_2 = \dfrac{p_1 + p_2}{2}$, which is the average of the probabilities p_1 and p_2. Therefore, $p > \frac{5}{7}$ implies that $p_1 > \frac{5}{7}$ or $p_2 > \frac{5}{7}$. To make an urn with $p_1 > \frac{5}{7}$ (say) we must have $(w, b) = (1, 0)$, $(2, 0)$, $(3, 0)$, $(3, 1)$, $(4, 0)$, or $(4, 1)$, where the urn contains w white balls and b black balls. These are the only distributions that could yield $p > \frac{5}{7}$. In the case of $(w, b) = (2, 0)$, $(3, 0)$, or $(4, 0)$ we have $p < \frac{5}{7}$, as moving all but one white ball to the other urn increases p_2 but leaves $p_1 = 1$. Finally, $(w, b) = (3, 1)$ or $(4, 1)$ yields $p = \frac{1}{2}$ or $p = \frac{2}{5}$, each less than $\frac{5}{7}$.

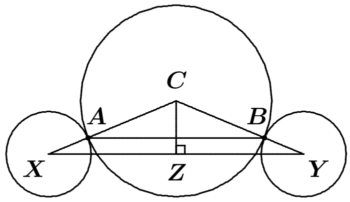
Our first distribution maximizes our chance of obtaining a white ball.

4. The three attached barrels. Three big cylindrical barrels, lying parallel to the earth, are attached by a steel cable at their contact points, A and B, such that they stay fixed in place. Knowing that the two smaller ones each have a radius of 4 metres and the biggest one has a radius of 9 metres, what is the length of the steel cable?

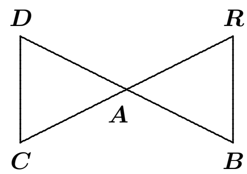


Solution by the editor.

Join the centres of the smaller barrels and drop a perpendicular to this segment from the centre of the larger barrel, as in the diagram at right. Since the barrels rest on the earth, the length of CZ is the difference of their radii, that is, $|CZ| = 9 - 4 = 5$. Also the length of CY is the sum of the radii, that is, $|CY| = 9 + 4 = 13$. By the Pythagorean Theorem $|ZY| = \sqrt{13^2 - 5^2} = 12$. Thus, $|XY| = 24$. Finally, since triangle CAB is similar to triangle CXY, we have $|AB| = \frac{9}{13}|XY| = \frac{216}{13}$.



5. The magic words. An illusionist is searching for magic words to accompany his many magic tricks. He decides to construct his magic words starting with the diagram on the right. He takes a path through the diagram and jots downs the letters he finds on it. Each magic word must have exactly 11 letters and must start and end with the letter A. Two consecutive letters must never be identical. How many magic words are there?



Note: Here are two possible magic words: $ABRACADABRA$ and $ARADCABARBA$.

Solution by the editor.

Let M_k be the number of k-letter words that start with A and end with A and that can be formed by travelling through the bowtie. Let N_k be the number of k-letter words starting with A but *not* ending with the letter A that can be similarly formed.

If $k \geq 2$, then we see that $M_k = N_{k-1}$, because removing the letter A from a word ending with A leaves a word not ending in A (but still starting with A) and the process can be reversed. Similarly, by deleting the last letter of a word of length k that starts with A but does not end in A, we see that $N_k = N_{k-1} + 4M_{k-1}$, because any of the four letters different from A can be added to a word not ending in A or else there is only way to extend a $(k-1)$-letter word not ending in A to one that still does not end in A. Since $M_{k-1} = N_{k-2}$ the last equation becomes $N_k = N_{k-1} + 4N_{k-2}$, where $k \geq 1$.

We now have $N_1 = 0$, $N_2 = 4$, $N_3 = N_2 + 4N_1 = 4 + 4 \cdot 0 = 4$, and so forth. The results of calculating the N_i are summarized in the table below:

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}
0	4	4	20	36	116	260	724	1764	4660

Finally $M_{11} = N_{10}$, so there are 4660 magic words altogether.

6. **All ten digits**. Find the smallest positive natural number N such that, in the decimal notation, N and $2N$ together use all ten digits: $0, 1, 2, \ldots, 9$.

Solution by the editor.

We have $2(13485) = 26970$, and we will prove that if N_1 and $2N_1$ together use all ten digits and $N_1 \leq N = 13485$, then $N_1 = N$.

As N_1 has five digits and $N_1 \leq N$, then $N_1 = 1\ldots$ and $2N_1 = 2\ldots$. Digits 1 and 2 are now used and $2N_1$ uses 0 (otherwise N_1 uses 0 and $2N_1$ then uses 0 or 1, a contradiction). Thus, $N_1 = 13\ldots$. The smallest available digit for N_1 is now a 4 and $N_1 \leq N$, hence $N_1 = 134\ldots$ and $2N_1 = 26\ldots$. The number N_1 uses 5, because $2N_1$ uses 0. If $N_1 = 1345x$, then x is a digit greater than 5 and $2N_1 = 2691y$, a contradiction. Thus, $N_1 = 134x5 \leq N$. Finally, $x \neq 7$, hence $x = 8$.

Therefore, $N_1 = N$ and N is the smallest positive integer with the given property.

[*Ed.:* Rolland Gaudet offers the solution $N = 6792$ if initial zeroes are allowed, for then $2(6792) = 013584$.]

7. **The pizza toppings**. At the Julio pizzeria, all the pizzas have cheese and tomato sauce on them. The choice of toppings is limited to black olives, anchovies, and sausage. Of the 200 clients Julio had yesterday, 40 took anchovies, 80 took black olives, 120 took sausage, 60 took at the same time black olives and sausage, but none took at the same time anchovies and black olives or anchovies and sausage. How many clients took none of the three toppings?

Solution by the editor.

Let t be the number of customers who took at least one topping. Any customer who took anchovies took no other topping, so $t = 40 + x$ where x is the number of customers who took black olives or sausage (or both). There were 60 customers who took both black olives and sausage, so $20 = 80 - 60$ took just black olives and nothing else. Similarly, $60 = 120 - 60$ customers took sausage and nothing else. Thus, $t = 40 + x = 40 + (20 + 60 + 60) = 180$, and the number of customers who took no toppings is $200 - t = 20$.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga) and Eric Robert (Leo Hayes High School, Fredericton).

Mayhem Problems

Please send your solutions to the problems in this edition by **1 May 2009**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M376. *Proposed by the Mayhem Staff.*

Determine the value of x if $\left(10^{2009}+25\right)^2-\left(10^{2009}-25\right)^2=10^x$.

M377. *Proposed by the Mayhem Staff.*

An arithmetic sequence consists of 9 positive integers. The sum of the terms in the sequence is greater than 200 and less than 220. If the second term in the sequence is 12, determine the sequence.

M378. *Proposed by the Mayhem Staff.*

Points C and D are chosen on the semi-circle with diameter AB so that C is closer to A. Segments CB and DA intersect at P; segments AC and BD extended intersect at Q. Prove that QP extended is perpendicular to AB.

M379. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The integers $27+C$, $555+C$, and $1371+C$ are all perfect squares, the square roots of which form an arithmetic sequence. Determine all possible values of C.

M380. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Triangle ABC is right-angled at C and has $BC = a$ and $CA = b$, with $a \geq b$. Squares $ABDE$, $BCFG$, and $CAHI$ are drawn externally to triangle ABC. The lines through FI and EH intersect at P, the lines through FI and DG intersect at Q, and the lines through DG and EH intersect at R. If triangle PQR is right-angled, determine the value of $\frac{b}{a}$.

M381. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all solutions to the equation

$$\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-6} + \frac{1}{x-7} = x^2 - 4x - 4\,.$$

....

M376. *Proposé par l'Équipe de Mayhem.*

Déterminer la valeur de x si $\left(10^{2009} + 25\right)^2 - \left(10^{2009} - 25\right)^2 = 10^x$.

M377. *Proposé par l'Équipe de Mayhem.*

Déterminer la suite arithmétique formée de 9 entiers positifs dont la somme se situe entre 200 et 220 et dont le second terme vaut 12.

M378. *Proposé par l'Équipe de Mayhem.*

A partir du point A, on choisit deux points C et D sur un demi-cercle de diamètre AB. Soit P l'intersection des droites CB et DA, et Q celle des droites AC et BD. Montrer que la droite PQ est perpendiculaire à AB.

M379. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Les entiers $27 + C$, $555 + C$ et $1371 + C$ sont tous des carrés parfaits dont les racines carrées forment une suite arithmétique. Trouver toutes les valeurs possibles de C.

M380. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans un triangle ABC d'angle droit en C, soit $BC = a$ et $CA = b$, avec $a \geq b$. Extérieurement au triangle ABC, on construit les carrés $ABDE$, $BCFG$ et $CAHI$. Soit respectivement P, Q et R les intersections des droites FI et EH, FI et DG, DG et EH. Déterminer la valeur de $\frac{b}{a}$ pour que PQR soit un triangle rectangle.

M381. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Déterminer toutes les solutions de l'équation

$$\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-6}+\frac{1}{x-7} = x^2-4x-4\,.$$

Mayhem Solutions

M338. *Proposed by the Mayhem Staff.*

Two students miscopy the quadratic equation $x^2 + bx + c = 0$ that their teacher writes on the board. Jim copies b correctly but miscopies c; his equation has roots 5 and 4. Vazz copies c correctly, but miscopies b; his equation has roots 2 and 4. What are the roots of the original equation?

Solution by Taylor Thetford, student, Lakeview High School, San Angelo, TX, USA.

The roots of the quadratic equation that Jim writes down are 5 and 4. His quadratic equation is thus $(x-5)(x-4) = x^2-9x+20 = 0$. Since Jim copied b correctly, we can conclude that in the original quadratic equation, $b = -9$.

Similarly, since Vazz's roots are 2 and 4, his quadratic equation has the form $(x-2)(x-4) = x^2-6x+8 = 0$. Since Vazz copied c correctly, then $c = 8$.

Thus, the original equation was $x^2-9x+8 = 0$. Factoring, we obtain $(x-1)(x-8) = 0$. Therefore, the roots of the original equation are 1 and 8.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; JOHAN GUNARDI, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON; and TITU ZVONARU, Comănești, Romania.

M339. *Proposed by the Mayhem Staff.*

(a) Determine the number of integers between 100 and 199, inclusive, which contain *exactly* two equal digits.

(b) An integer between 1 and 999 is chosen at random, with each integer being equally likely to be chosen. What is the probability that the integer has *exactly* two equal digits?

Solutions by Taylor Thetford, student, Lakeview High School, San Angelo, TX, USA.

(a) Each of the integers in the given range can be written in the form $1xy$. There are three cases to consider.

Case 1. The first and last digits are the same. Here, we are looking for integers $1x1$ where the middle digit can take any value except 1. This yields 9 possibilities.

Case 2. The first and second digits are the same. Here, we are looking for integers $11y$ where the last digit can take any value except 1. This again yields 9 possibilities.

Case 3. The second and last digits are the same. Here, we are looking for integers $1xx$ where x is not 1. Again, there are 9 possibilities.

Adding the results from our three cases, we find that there are 27 numbers between 100 and 199, inclusive, that contain exactly two equal digits.

(b) We count the number of integers in the range 1 to 999, inclusive, that have exactly two equal digits.

First, between 1 and 99, there are 9 of these, namely, 11, 22, $\ldots$, 99. Next, between 100 and 199, we have counted 27 in part (a). Using the same argument as in (a), we can show that there are 27 numbers between 200 and 299, inclusive, and for every other interval of one hundred numbers up to the range of 900 to 999.

There are therefore $9 + 9 \cdot 27 = 252$ numbers between 1 and 999 which contain exactly two equal digits.

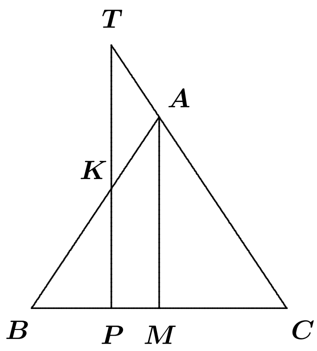
The probability that a randomly selected integer between 1 and 999 has exactly two equal digits is thus $\frac{252}{999} = \frac{28}{111}$.

Also solved by PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON. There were 3 incorrect solutions and 1 partial solution submitted.

M340. *Proposed by the Mayhem Staff.*

Let ABC be an isosceles triangle with $AB = AC$, and let M be the mid-point of BC. Let P be any point on BM. A perpendicular is drawn to BC at P, meeting BA at K and CA extended at T. Prove that $PK + PT$ is independent of the position of P (that is, the value of $PK + PT$ is always the same, no matter where P is placed).

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.



Since $\triangle ABC$ is isosceles with sides AB and AC of equal length, we have $MA \perp BC$. Also, since $PT \perp BC$, then $MA||PT$.

Since $MA||PK$, then $\triangle MBA$ is similar to $\triangle PBK$ since each is right-angled and they share the angle at B. From this, we obtain $\dfrac{PK}{PB} = \dfrac{MA}{MB}$, hence $PK = \dfrac{PB \cdot MA}{MB}$.

Similarly, since $MA||PT$, then $\triangle CPT$ is similar to $\triangle CMA$, whence $\dfrac{PT}{PC} = \dfrac{MA}{MC}$ and so $PT = \dfrac{PC \cdot MA}{MC}$.

Since $MB = MC = \frac{1}{2}BC$, we can conclude that

$$\begin{aligned} PK + PT &= \frac{PB \cdot MA}{MB} + \frac{PC \cdot MA}{MC} = \frac{(PB + PC) \cdot MA}{MB} \\ &= \frac{BC \cdot MA}{MB} = 2MA\,. \end{aligned}$$

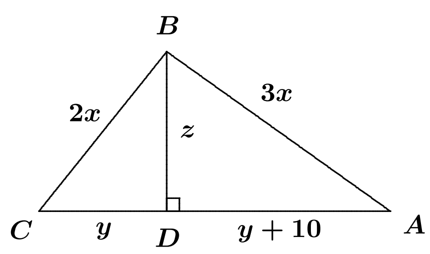
Thus, $PK + PT$ is independent of the position of P, since it depends only on the length of MA.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; and TITU ZVONARU, Comăneşti, Romania. There was 1 incorrect solution submitted.

M341. *Proposed by the Mayhem Staff.*

Let ABC be a right triangle with right angle at B. Sides BA and BC are in the ratio $3 : 2$. Altitude BD divides CA into two parts that differ in length by 10. What is the length of CA?

Solution by Taylor Thetford, student, Lakeview High School, San Angelo, TX, USA.



Let $2x$ and $3x$ be the lengths of CB and AB, respectively. Let y and $y + 10$ be the lengths of CD and DA, respectively. Let z be the length of BD. We wish to find $2y + 10$, which is the length of CA.

By applying the Pythagorean Theorem in $\triangle ABC$, we find that $(2x)^2 + (3x)^2 = (2y + 10)^2$ and so $13x^2 = 4y^2 + 40y + 100$.

Applying the Pythagorean Theorem to $\triangle BDC$ and $\triangle BDA$, we find that $y^2 + z^2 = 4x^2$ and $z^2 + (y+10)^2 = 9x^2$.

Eliminating z in the last two equations gives $4x^2 - y^2 = 9x^2 - (y+10)^2$. Therefore, $5x^2 = (y+10)^2 - y^2 = 20y + 100$ or $x^2 = 4y + 20$, and so $13x^2 = 52y + 260$.

Combining this result with $13x^2 = 4y^2 + 40y + 100$, we find that

$$\begin{aligned} 52y + 260 &= 4y^2 + 40y + 100\,; \\ 4y^2 - 12y - 160 &= 0\,; \\ y^2 - 3y - 40 &= 0\,; \\ (y+5)(y-8) &= 0\,. \end{aligned}$$

Since $y > 0$, then $y = 8$, and so $CA = 2y + 10 = 26$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON; and TITU ZVONARU, Comănești, Romania. There was 1 incorrect solution submitted.

M342. *Proposed by the Mayhem Staff.*

Quincy and Celine have to move 10 small boxes and 10 large boxes. The chart below indicates the time that each person takes to move each type of box.

	Celine	Quincy
small box	1 min.	3 min.
large box	6 min.	5 min.

They start moving the boxes at 9:00 am. What is the earliest time at which they can be finished moving all of the boxes?

Solution by Mayhem Staff.

Let x represent the number of small boxes and y represent the number of large boxes that Celine moves. Since there are 10 small boxes and 10 large boxes, then Quincy moves $10 - x$ small boxes and $10 - y$ large boxes.

Given the lengths of time that each takes, it takes Celine $x + 6y$ minutes and it takes Quincy $3(10-x) + 5(10-y) = 80 - 3x - 5y$ minutes. If $x = 9$ and $y = 4$, then Celine takes 33 minutes and Quincy takes 33 minutes. We show that it cannot be done faster than this.

If Quincy and Celine finish in fewer than 33 minutes, then each takes at most 32 minutes, so the total working time is at most 64 minutes, so $x + 6y + (80 - 3x - 5y) = 80 - 2x + y \le 64$ or $2x - y \ge 16$.

Since x and y are nonnegative integers and each is less than 10, then the possible pairs (x, y) that satisfy this inequality are $(8, 0)$, $(9, 0)$, $(9, 1)$, $(9, 2)$, $(10, 0)$, $(10, 1)$, $(10, 2)$, $(10, 3)$, and $(10, 4)$.

Since we want each of Celine's time and Quincy's time to be at most 32 minutes, then we need $x + 6y \le 32$ and $80 - 3x - 5y \le 32$. The first inequality eliminates the pair $(10, 4)$ from the list of possible pairs. The second inequality simplifies to $3x + 5y \ge 48$; none of the remaining pairs satisfy this inequality.

Thus, none of these possibilities take any less time than 33 minutes. Therefore, the earliest possible finishing time is 9:33 a.m.

There were 4 incorrect and 3 incomplete solutions submitted.

An expanded treatment of a similar problem appeared in the Problem of the Month column in ***CRUX with MAYHEM****, volume 34, number 2.*

M343. *Proposed by the Mayhem Staff.*

The Fibonacci numbers are defined by $f_1 = f_2 = 1$ and, for $n \ge 2$, by $f_{n+1} = f_n + f_{n-1}$. The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, $\ldots$. Find the sum of the first 100 even Fibonacci numbers.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Since $f_1 = f_2 = 1$, $f_3 = 2$, and $f_n = f_{n-1} + f_{n-2}$, then f_m is even if and only if m is a multiple of 3. (This is because the parities of the terms will form the pattern Odd, Odd, Even, Odd, Odd, Even, and so on.)

If $S_n = \sum\limits_{k=1}^{n} f_{3k}$, then

$$S_n \;=\; \frac{1}{2}\sum_{k=1}^{n}(f_{3k}+f_{3k}) \;=\; \frac{1}{2}\sum_{k=1}^{n}\big((f_{3k-2}+f_{3k-1})+f_{3k}\big) \;=\; \frac{1}{2}\sum_{k=1}^{3n} f_k\,. \quad (1)$$

Next, we have $f_1 = f_3 - f_2$, and $f_2 = f_4 - f_3$, and also $f_3 = f_5 - f_4$, and so on until $f_{r-1} = f_{r+1} - f_r$ and $f_r = f_{r+2} - f_{r+1}$.

Since the right side of the sum of the n equations above "telescopes", it follows that

$$\sum_{k=1}^{r} f_k \;=\; f_{r+2} - f_2 \;=\; f_{r+2} - 1\,. \qquad (2)$$

From (1) and (2), we find that $S_n = \frac{1}{2}(f_{3n+2} - 1)$. In our particular case, $S_{100} = \frac{1}{2}(f_{302} - 1)$. Maple computes the value of S_{100} to be exactly 290905784918002003245752779317049533129517076702883498623284700.

For the record, by Binet's formula for Fibonacci numbers we have that $f_m = \dfrac{1}{\sqrt{5}}\left(\alpha^m - \beta^m\right)$, where $\alpha = \dfrac{1}{2}(1+\sqrt{5})$ and $\beta = \dfrac{1}{2}(1+\sqrt{5})$. Hence the required sum is also given by $S_{100} = \dfrac{1}{2\sqrt{5}}\left(\alpha^{302} - \beta^{302}\right) - \dfrac{1}{2}$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; DIVYANSHU RANJAN, Delhi, India; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; and TITU ZVONARU, Comănești, Romania.

Problem of the Month

Ian VanderBurgh

Approximation is one of the most important concepts in mathematics.

Problem (2006 Canadian Open Mathematics Challenge) Determine, with justification, the fraction $\frac{p}{q}$, where p and q are positive integers and $q < 100$, that is closest to, but not equal to, $\frac{3}{7}$.

While it is tempting to get out your calculator, it can initially only help so much. If you calculate $\frac{3}{7}$, you'll obtain $0.428571\ldots$. This doesn't help in any obvious way to answer the question.

A first approach after the calculator might be to go for the fraction with the largest possible denominator. This makes a lot of sense in some ways, as the fractions with the largest denominators will be closest together and so would seem to have the best chance of being closest to $\frac{3}{7}$. In our case, the largest possible denominator is $q = 99$. The given fraction, $\frac{3}{7}$, is between $\frac{42}{99} = 0.424242\ldots$ and $\frac{43}{99} = 0.434343\ldots$. After a quick look, we can tell that $\frac{3}{7}$ is closer to $\frac{42}{99}$. From the decimal approximations, $\frac{3}{7}$ and $\frac{42}{99}$ differ by about 0.004. Is this the closest of all possible fractions?

Another idea is to try to convert $\frac{3}{7}$ into the equivalent fraction with the largest possible denominator and then adjust from there. Multiplying numerator and denominator by 14, we obtain $\frac{42}{98}$. We could then add 1 or -1 to the numerator to obtain $\frac{41}{98}$ or $\frac{43}{98}$, which differ from $\frac{3}{7}$ by $\frac{1}{98}$. But this means that the difference is bigger than 0.01, which is worse than before, so this approach doesn't give a closer fraction.

Can we do better than $\frac{42}{99}$? It is possible that, even though fractions with smaller denominators are further apart, they can be between some of the other fractions that we've looked at, for example between $\frac{42}{99}$ and $\frac{3}{7}$ or between $\frac{43}{99}$ and $\frac{3}{7}$.

Solution We want to use the fraction $\frac{p}{q}$ to approximate $\frac{3}{7}$. Let's calculate their difference, which is what we want to minimize:

$$\left|\frac{p}{q} - \frac{3}{7}\right| = \left|\frac{7p - 3q}{7q}\right| = \frac{|7p - 3q|}{7q}.$$

What can we do to make this as small as possible? Two approaches would be to make the numerator of the difference as small as possible or to make the denominator of the difference as large as possible.

Let's focus initially on the numerator. The numerator cannot equal 0 because the fractions $\frac{p}{q}$ and $\frac{3}{7}$ are not equal. Thus, the smallest possible value for the numerator is 1, because p and q are integers. So let's try to find values of p and q for which the numerator equals 1. In this case, the difference equals $\frac{1}{7q}$ which is minimized when q is largest.

For the numerator to equal 1, we need $7p - 3q = \pm 1$. Since we also want to maximize q, we rewrite this as $7p = 3q \pm 1$ and work from the largest possible integer values of q to see when we also get an integer value for p.

If $q = 99$, the equation becomes $7p = 3(99) \pm 1 = 297 \pm 1$. Neither possibility is a multiple of 7.

If $q = 98$, the equation becomes $7p = 3(98) \pm 1 = 294 \pm 1$. Neither possibility is a multiple of 7.

If $q = 97$, the equation becomes $7p = 3(97) \pm 1 = 291 \pm 1$. Neither possibility is a multiple of 7.

If $q = 96$, the equation becomes $7p = 3(96) \pm 1 = 288 \pm 1$. Since 287 is a multiple of 7, then taking $q = 96$ and $p = 41$ gives a difference with numerator 1.

So we have $\left|\frac{41}{96} - \frac{3}{7}\right| = \frac{1}{7 \cdot 96} = \frac{1}{672}$ and this is the smallest possible difference with the numerator equal to 1.

If the numerator equalled 2 or something larger, then the smallest possible difference occurs when the numerator is as small as possible and the denominator is as large as possible, so is $\frac{2}{7 \cdot 99} = \frac{2}{693}$. This is the smallest possible difference with numerator at least 2.

Combining the cases, the smallest possible difference is indeed $\frac{1}{672}$, and so the closest fraction to $\frac{3}{7}$ of all of the fractions under consideration is $\frac{p}{q} = \frac{41}{96}$.

The approximation of functions with polynomials is often seen in first-year university calculus courses. As part of these investigations, we learn how to estimate the amount of error when approximating, for example, $\sin x$ with $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$. The techniques used to estimate this type of error are not dissimilar to what we have seen above, and are very useful in many types of calculations.

THE OLYMPIAD CORNER

No. 275

R.E. Woodrow

The year has flown by, and it has brought many changes to Crux and to the *Corner*. Of course it has been overshadowed by the sudden and untimely loss of a great friend and a devoted colleague, Jim Totten, mid-way through the transition to a new Editor-in-Chief. I think Vazz Linek has done a wonderful job of stepping in and keeping the journal on track with only an understandable slowing of the production pace in the interim.

Readers will have noticed the announcement at the end of the December *Corner* that Joanne Canape, who has transformed my scribbles into a high quality tex file for many years, has decided that two decades is enough. As I customarily begin the year by thanking all those who contributed to the *Corner* in the last year, I would be very remiss not to lead off with sincere thanks to Joanne.

It is also appropriate to thank those who submitted problem sets for our use as well as a special thanks to the dedicated readers who furnish their nice solutions which we use. Hoping, as always, that I've not missed someone, here is the list for the 2008 members of the *Corner*.

Arkady Alt
Miguel Amengual Covas
Jean-Claude Andrieux
Houda Anoun
Ricardo Barroso Campos
Michel Bataille
José Luis Díaz-Barrero
J. Chris Fisher
Kipp Johnson
Geoffrey A. Kandall
Ioannis Katsikis
R. Laumen
Salem Malikic
Pavlos Maragoudakis
Robert Morewood
Andrea Munaro
Vedula N. Murty
Felix Recio
Xavier Ros
D.J. Smeenk
Babis Stergiou
Daniel Tsai
Panos E. Tsaoussoglou
George Tsapakidis
Jan Verster
Edward T.H. Wang
Luyan Zhong-Qiao
Li Zhou
Titu Zvonaru

Our apologies to Svetoslav Savchev for the misspelling of his name in the December 2008 Olympiad.

For your problem solving pleasure in the new year we start off with the problems of the German Mathematical Olympiad, Final Round, 2006. My thanks go to Robert Morewood, Canadian Team Leader to the 47^{th} IMO in Slovenia 2006, for collecting them for our use.

German Mathematical Olympiad
Final Round, Grades 12–13
Munich, April 29 – May 2, 2006

First Day

1. Determine all positive integers n for which the number

$$z_n = \underbrace{101\cdots101}_{2n+1 \text{ digits}}$$

is a prime.

2. Five points are on the surface of a sphere of radius 1. Let a_{min} denote the smallest distance (measured along a straight line in space) between any two of these points. What is the maximum value for a_{min}, taken over all arrangements of the five points?

3. Find all positive integers n for which the numbers 1, 2, 3, $\dots$, $2n$ can be coloured with n colours in such a way that every colour appears twice and every number 1, 2, 3, $\dots$, n appears exactly once as the difference of two numbers with the same color.

Second Day

4. Let D be a point inside the triangle ABC such that $AC - AD \geq 1$ and $BC - BD \geq 1$. Prove that $EC - ED \geq 1$ for any point E on the side AB.

5. Let x be a nonzero real number satisfying the equation $ax^2 + bx + c = 0$. Furthermore, let a, b, and c be integers satisfying $|a| + |b| + |c| > 1$. Prove that

$$|x| \geq \frac{1}{|a| + |b| + |c| - 1}.$$

6. Let a circle through B and C of a triangle ABC intersect AB and AC in Y and Z, respectively. Let P be the intersection of BZ and CY, and let X be the intersection of AP and BC. Let M be the point that is distinct from X and on the intersection of the circumcircle of the triangle XYZ with BC. Prove that M is the midpoint of BC

Our second problem set for this number is a set of selected problems from the Thai Mathematical Olympiad Examinations 2005. Again, thanks go to Robert Morewood, team leader to the 47th IMO in Slovenia 2006, for collecting them for the *Corner*.

Thai Mathematical Olympiad Examinations 2005
Selected Problems

1. Let $P(x)$, $Q(x)$, and $R(x)$ be polynomials satisfying

$$2xP(x^3) \ + \ Q(-x-x^2) \ = \ (1+x+x^2)\,R(x)\,.$$

Show that $x-1$ is a factor of $P(x)-Q(x)$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y+f(xy)) \ = \ f(f(x+y)) \ + \ xy$$

for all x, $y \in \mathbb{R}$.

3. Let a, b, and c be positive real numbers. Prove that

$$1 \ + \ \frac{3}{ab+bc+ca} \ \geq \ \frac{6}{a+b+c}\,.$$

4. Let n be a positive integer. Prove that $n(n+1)(n+2)$ is not a perfect square.

5. Find the least positive integer n such that $2549 \mid \left(n^{2545} - 2541\right)$.

6. Do there exist positive integers x, y, and z such that

$$2548^x \ + \ (-2005)^y \ = \ (-543)^z\,?$$

7. Show that there exist positive integers m and n such that $\gcd(m,n) = 1$ and $2549 \mid ((25 \cdot 49)^m + 25^n - 2 \cdot 49^n)$.

8. The median AM of a triangle ABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y, respectively. The lines AX and AY intersect BC at P and Q. Prove that $BC = CQ$. (Shortlist 2005)

9. Let ABC be an acute-angled triangle with $AB \neq AC$, let H be its orthocentre and M the midpoint of BC. Points D on AB and E on AC are such that $AE = AD$ and D, H, and E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE. (Shortlist 2005)

10. Assume ABC is an isosceles triangle with $AB = AC$. Suppose that P is a point on the extension of side BC. X and Y are points on lines AB and AC such that $PX \| AC$ and $PY \| AB$. Let T be the midpoint of arc BC. Prove that $PT \perp XY$. (Iran 2004)

As a third set of problems we give the 46th Ukrainian Mathematical Olympiad Final Round 2006 - 11th form problems. Again, thanks go to Robert Morewood, team leader to the 47th IMO in Slovenia 2006, for collecting them for our use.

46th Ukrainian Mathematical Olympiad 2006
Final Round
11th Form

1. (V.V. Plakhotnyk) Prove that for any rational numbers a and b the graph of the function

$$f(x) \;=\; x^3 - 6abx - 2a^3 - 4b^3\,, \quad x \in \mathbb{R}$$

has exactly one point in common with the x-axis.

2. (O.A. Sarana) A circle is divided into 2006 equal arcs by 2006 points. Baron Munchausen claims that he can construct a closed polygonal curve with the set of vertices consisting of these 2006 points such that amongst its 2006 edges there are no two which are parallel to each other. Is his claim true or false?

3. (T.M. Mitelman)

(a) Prove that for any rational number $\alpha \in (0,1)$ there exists an infinite set of real numbers that satisfy the equation $\{x\lfloor x\{x\}\rfloor\} = \alpha$ and any two of them have the same fractional part. (The fractional part of a real number a is given by $\{a\} = a - \lfloor a\rfloor$, where $\lfloor a\rfloor$ is its integer part, that is, the greatest integer that does not exceed a.)

(b) Prove that for any rational number $\alpha \in (0,1)$ there exists an infinite set of real numbers that satisfy the equation $\{x\lfloor x\{x\}\rfloor\} = \alpha$ and any two of them have *different* fractional parts.

4. (V.A. Yasinskiy) Two circles ω_1 and ω_2 intersect each other at two distinct points A and B. The tangent line of the circle ω_1 at the point A and the tangent line of the circle ω_2 at the point B meet at point C. The first of these two lines intersects the circle ω_2 for the second time at point $T \neq A$. The point X (distinct from A and B) is on the circle ω_1, and the line XA intersects the circle ω_2 for the second time at point Y (distinct from A). The lines YB and XC meet at point Z. Prove that TZ is parallel to XY.

5. (O.O. Kurchenko) Prove that for any real numbers x and y

$$|\cos x| \;+\; |\cos y| \;+\; |\cos(x+y)| \;\geq\; 1\,.$$

6. (T.M. Mitelman) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^3 + y^3\right) \;=\; x^2 f(x) \;+\; y f(y^2)$$

for all real numbers x and y.

7. (V.A. Yasinskiy) A point M lies inside a cube $ABCDA_1B_1C_1D_1$. Points A', B', C', D', A_1', B_1', C_1', and D_1' belong to the rays MA, MB, MC, MD, MA_1, MB_1, BC_1, and MD_1 respectively. Prove that if the polyhedron $A'B'C'D'A_1'B_1'C_1'D_1'$ is a parallelepiped (that is, all of its faces are parallelograms), then it is a cube.

8. (V.A. Yasinskiy) There are $n \geq 3$ soldiers in captain Petrenko's squad, no two of the same height. The captain orders them to stand single-file (not necessarily sorted by height). A "wave" is any subsequence of (not necessarily next to each other) soldiers in this line such that the first (leftmost) soldier in the wave is taller than the second soldier in it, but the second soldier in it is shorter than the third one, who is in turn taller than the fourth one, and so on. (For example, if $n = 9$, the soldiers are enumerated by height, and the captain lines them up as 9, 3, 5, 7, 1, 2, 6, 4, 8 then a longest wave for this line-up is 9, 3, 7, 1, 6, 4, 8. However, if the captain lines them up as 1, 2, 3, 4, 5, 6, 7, 8, 9, then a longest wave consists of (any) one soldier.) For every n, consider the number of possible lines with the longest waves of even lengths and the number of possible lines with the longest waves of odd lengths. Which of these numbers is bigger?

Continuing with problems for readers to solve we give the Czech-Polish-Slovak Mathematics Competition written on June 26-28, 2006 at Žilina, Slovakia. Thanks again go to Robert Morewood, Canadian team leader to the 47th IMO in Slovenia 2006, for collecting them for our use.

Czech-Polish-Slovak Mathematics Competition 2006

1. Five distinct points A, B, C, D, and E lie in this order on a circle of radius r and satisfy $AC = BD = CE = r$. Prove that the orthocentres of the triangles ACD, BCD, and BCE are the vertices of a right-angled triangle.

2. There are n children sitting at a round table. Erika is the oldest among them and she has n candies. No other child has any candy. Erika distributes the candies as follows. In every round, all the children with at least two candies show their hands. Erika chooses one of them and he/she gives one candy to each of the children sitting next to him/her. (So in the first round Erika must choose herself to begin the distribution.) For which $n \geq 3$ is it possible to redistribute the candies so that each child has exactly one candy?

3. The sum of four real numbers is 9 and the sum of their squares is 21. Prove that these four numbers can be labelled as a, b, c, and d so that the inequality $ab - cd \geq 2$ holds.

4. Prove that for every positive integer k there is a positive integer n such that the decimal representation of 2^n has a block of exactly k consecutive zeros, that is, $2^n = \cdots a00\cdots 0b\cdots$, where a and b are nonzero digits with k zeros between them.

5. Find the number of integer sequences $(a_n)_{n=1}^{\infty}$ such that $a_n \neq -1$ and

$$a_{n+2} = \frac{a_n + 2006}{a_{n+1} + 1}$$

for every positive integer n.

6. Is there a convex pentagon $A_1A_2\ldots A_5$ such that for each i the lines A_iA_{i+3} and $A_{i+1}A_{i+2}$ intersect in B_i and the points B_1, B_2, $\ldots$, B_5 are collinear? (By convention $A_6 = A_1$, $A_7 = A_2$, and $A_8 = A_3$.)

Our final problem set for this issue is the XXI Olimpiadi Italiano della Matematica, Cesenatico, written 5 May 2006. Thanks again go to Robert Morewood, Canadian team leader to the 47th IMO in Slovenia, for collecting them for our use.

XXI Olimpiadi Italiano della Matematica
Cesenatico
May 5, 2006

1. Rose and Savino play a game with a deck of traditional Neapolitan playing cards which consists of 40 cards of four different suits, numbered 1 to 10. At the start each player has 20 cards. Taking turns, one shows a card on the table. Whenever some cards on the table add to exactly 15, these are then removed from the game (if the sum 15 can be obtained in more than one way, the player who last moved decides which cards adding to 15 to remove). At the end of the game only one card, a 9, is left on the table. Savino holds two cards numbered 3 and 5, and Rose holds one card. What is the number of Rose's card?

2. Find all values of m, n, and p such that

$$p^n + 144 = m^2\,,$$

where m and n are positive integers and p is a prime number.

3. Let A and B be two points on a circle Γ such that AB is not a diameter. Let P be a point on Γ different from A and B, and let H be the orthocentre of the triangle ABP. Find the locus of H as P varies over all points of Γ different from A and B.

4. On an infinite chessboard all the positive integers are written in ascending order along a spiral, starting from 1 and proceeding anticlockwise; a portion of the chessboard is shown in the figure.

A "right half-line" of the chessboard is the set of squares given by a square C and by all squares in the same row as C and to the right of C.

17	16	15	14	13
18	5	4	3	12
19	6	1	2	11
20	7	8	9	10
21	22	23	24	25

(a) Prove that there exists a right half-line none of whose squares contains a multiple of 3.

(b) Determine if there exist infinitely many pairwise disjoint right half-lines none of whose squares contains a multiple of 3.

5. Consider the inequality

$$(x_1 + \cdots + x_n)^2 \;\geq\; 4(x_1x_2 + x_2x_3 + \cdots + x_nx_1)\,.$$

(a) Determine for which $n \geq 3$ the inequality holds true for all possible choices of positive real numbers x_1, x_2, $\ldots$, x_n.

(b) Determine for which $n \geq 3$ the inequality holds true for all possible choices of *any* real numbers x_1, x_2, $\ldots$, x_n.

6. Albert and Barbara play a game. At the start there are some piles of coins on the table, not all necessarily with the same number of coins. The players move in turn and Albert starts. At each turn a player may either take a coin from a pile ***or*** divide a pile into two piles with each pile containing at least one coin (a player may exercise only one of these options).

The one who takes the last coin wins the game. In terms of the number of piles and the number of coins in each pile at the start, determine which of the players has a winning strategy.

Now we turn to our file of solutions from the readers to problems from the March 2008 number of the *Corner* and the Estonian IMO Selection Contest 2004-2005, given at [2008 : 79-80].

3. Find all pairs (x, y) of positive integers satisfying $(x + y)^x = x^y$.

Solution by Konstantine Zelator, University of Toledo, Toledo, OH, USA.

We show that there are exactly two such pairs, $(x, y) = (2, 6)$, $(3, 6)$. We will make use of two basic facts from elementary number theory.

(a) If $a,\, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$, then $\gcd(a^m, b^n) = 1$ for any positive integers m and n.

(b) If $a|b$, a is a positive integer, $b \in \mathbb{Z}$, and $\gcd(a, b) = 1$, then $a = 1$.

Let x and y be positive integers that satisfy $(x + y)^x = x^y$. Write $d = \gcd(x, y)$ and let $x = dx_1$, $y = dy_1$, where x_1 and y_1 are positive integers that are relatively prime. Substituting for x and y yields

$$d^{x_1 d} \cdot (x_1 + y_1)^{x_1 d} \;=\; x_1^{y_1 d} \cdot d^{y_1 d}\,. \tag{1}$$

We make cases by comparing the sizes of x_1 and y_1.

Case 1. Suppose that $x_1 = y_1$. Since $\gcd(x_1, y_1) = 1$, we have $x_1 = y_1 = 1$. Thus, equation (1) becomes $d^d \cdot 2^d = d^d$, which is impossible since $d \geq 1$.

Case 2. Suppose that $y_1 < x_1$. Then $x_1 - y_1$ is a positive integer and from equation (1) we obtain

$$d^{d(x_1 - y_1)} \cdot (x_1 + y_1)^{x_1 d} \;=\; x_1^{y_1 d}\,. \tag{2}$$

Since $\gcd(x_1, y_1) = 1$ it follows that $\gcd(x_1 + y_1, x_1) = 1$. By (a) above, we have $\gcd\big((x_1 + y_1)^{x_1 d}, x_1^{y_1 d}\big) = 1$. However, by equation (2), the positive integer $(x_1 + y_1)^{x_1 d}$ is a divisor of $x_1^{y_1 d}$. Since these two integers are relatively prime, it follows by (b) that $(x_1 + y_1)^{x_1 d} = 1$, which is impossible since $x_1 + y_1 \geq 2$ and $x_1 \cdot d \geq 1$.

Case 3. Suppose that $x_1 < y_1$. From (1) we obtain

$$(x_1 + y_1)^{x_1 d} \;=\; x_1^{y_1 d} \cdot d^{d(y_1 - x_1)}\,. \tag{3}$$

Since $\gcd(x_1, y_1) = 1$ we have $\gcd\big((x_1 + y_1)^{x_1 d}, x_1^{y_1 d}\big) = 1$ and from equation (3) we see that $x_1^{y_1 d}$ is a divisor of $(x_1 + y_1)^{x_1 d}$, which implies that $x_1^{y_1 d} = 1$. Since $y_1 d$ is a positive integer this means that $x_1 = 1$. Going back to equation (3) we see that $(1 + y_1)^d = d^{d(y_1 - 1)}$, hence

$$1 + y_1 \;=\; d^{y_1 - 1}\,. \tag{4}$$

Note that $d \neq 1$; otherwise equation (4) becomes $1 + y_1 = 1$, contrary to the fact that y_1 is a positive integer. Thus, $d \geq 2$. Since $y_1 = 1$ does not satisfy equation (4), we also have $y_1 \geq 2$. Setting $k = y_1 - 1$ equation (4) then becomes $k + 2 = d^k$. By Induction (or the Binomial Theorem) we obtain $2^k > k + 2$ for all integers $k \geq 3$. Since $d^k \geq 2^k$, it follows from $k + 2 = d^k$ that $k = 1$ or $k = 2$.

For $k = 2$ we have $4 = d^2$, hence $d = 2$. From $2 = k = y_1 - 1$ we then have $y_1 = 3$. Recall that $x_1 = 1$. Going back, we have $x = x_1 d = 1 \cdot 2 = 2$ and $y = y_1 d = 3 \cdot 2 = 6$. This is the solution $(x, y) = (2, 6)$.

Similarly, for $k = 1$ we have $d = 3$. Then $y_1 = k + 1 = 2$ and since $x_1 = 1$ we obtain $x = dx_1 = 3 \cdot 1 = 3$ and $y = dy_1 = 3 \cdot 2 = 6$. This is the other solution $(x, y) = (3, 6)$.

4. Find all pairs (a, b) of real numbers such that all roots of the polynomials $6x^2 - 24x - 4a$ and $x^3 + ax^2 + bx - 8$ are non-negative real numbers.

Solution by Titu Zvonaru, Comăneşti, Romania.

Let β_1, β_2, and β_3 be the roots of the polynomial $x^3 + ax^2 + bx - 8$, so that $x^3 + ax^2 + bx - 8 = (x - \beta_1)(x - \beta_2)(x - \beta_3)$. Comparing coefficients yields $\beta_1 + \beta_2 + \beta_3 = -a$ and $\beta_1\beta_2\beta_3 = 8$. Since β_1, β_2, and β_3 are nonnegative real numbers, by the AM-GM Inequality we have

$$\beta_1 + \beta_2 + \beta_3 \geq 3\sqrt[3]{\beta_1\beta_2\beta_3}\,,$$

hence $-a \geq 6$ or $a \leq -6$. The equation $6x^2 - 24x - 4a = 0$ has real roots if and only if $24^2 - 4 \cdot (-4a) \cdot 6 \geq 0$, which implies $24^2 + 24 \cdot 4a \geq 0$ and hence $a \geq -6$. Therefore, $a = -6$.

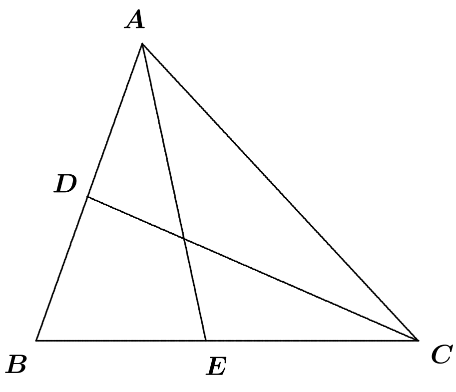
Now we have $\beta_1 + \beta_2 + \beta_3 = 6 = 3\sqrt[3]{\beta_1\beta_2\beta_3}$, from which it follows that $\beta_1 = \beta_2 = \beta_3 = 2$ and $b = 12$. Thus, the only pair satisfying the condition is $(a, b) = (-6, 12)$.

Next we turn to a solution to a problem of the Trentième Olympiad Mathématique Belge Maxi Finale, Mercredi 20 avril 2005 given at [2008 : 80].

3. Dans le triangle ABC, les droites AE et CD sont les bissectrices intérieures des angles $\angle BAC$ et $\angle ACB$ respectivement ; E appartient à BC et D appartient à AB. Pour quelles amplitudes de l'angle $\angle ABC$ a-t-on certainement

(a) $|AD| + |EC| = |AC|$? (b) $|AD| + |EC| > |AC|$?
(c) $|AD| + |EC| < |AC|$?

Solution by Titu Zvonaru, Comăneşti, Romania



As usual let $a = BC$, $b = CA$, and $c = AB$. The first equation below is the Angle Bisector Theorem; the following equations are equivalent to it:

$$\begin{aligned} \frac{BE}{EC} &= \frac{AB}{AC}\,; \\ \frac{BE}{EC} &= \frac{c}{b}\,; \\ \frac{BE + EC}{EC} &= \frac{b + c}{b}\,; \\ EC &= \frac{ab}{b + c}\,. \end{aligned}$$

Similarly, $AD = \dfrac{bc}{a + b}$.

We have

$$|AD| + |EC| - |AC| = \frac{bc}{a+b} + \frac{ab}{b+c} - b$$
$$= b\left(\frac{c(b+c)+a(a+b)-(a+b)(b+c)}{(a+b)(b+c)}\right).$$

By the Law of Cosines $a^2+c^2-b^2 = 2ac\cos B$, so the equation above can be rewritten as

$$|AD| + |EC| - |AC| = \frac{2abc\left(\cos(B)-\frac{1}{2}\right)}{(a+b)(b+c)}.$$

Hence,

$$|AD| + |EC| = |AC| \iff \angle ABC = 60^\circ,$$
$$|AD| + |EC| > |AC| \iff \angle ABC < 60^\circ,$$
$$|AD| + |EC| < |AC| \iff \angle ABC > 60^\circ.$$

Next we turn to solutions from our readers to problems of the 2005 Vietnam Mathematical Olympiad given at [2008 : 81].

1. Find the smallest and largest values of the expression $P = x + y$, where x and y are real numbers satisfying $x - 3\sqrt{x+1} = 3\sqrt{y+2} - y$.

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give the solution of Bataille.

We show that $P_{\min} = \frac{1}{2}\left(9+3\sqrt{21}\right)$ and $P_{\max} = 9+3\sqrt{15}$.

First, $x = -1$ and $y = \frac{1}{2}\left(11+3\sqrt{21}\right)$ satisfy the constraint equation $x-3\sqrt{x+1} = 3\sqrt{y+2}-y$ (easily checked using $10+2\sqrt{21} = \left(\sqrt{3}+\sqrt{7}\right)^2$) and $P = \frac{1}{2}\left(9+3\sqrt{21}\right)$. Similarly, for $x = \frac{1}{2}\left(10+3\sqrt{15}\right)$, $y = \frac{1}{2}\left(8+3\sqrt{15}\right)$, we have $P = 9+3\sqrt{15}$ and the constraint is satisfied. Now, let x and y satisfy the constraint equation. Then $P = 3\sqrt{x+1}+3\sqrt{y+2}$, so that

$$P^2 = 9\left(P+3+2\sqrt{(x+1)(y+2)}\right). \tag{1}$$

It follows that $P \geq 0$ and $P^2 - 9P - 27 \geq 0$. Thus, P is not less than the positive solution of the quadratic $x^2 - 9x - 27 = 0$ and we deduce that $P \geq \frac{1}{2}\left(9+3\sqrt{21}\right)$. From the AM-GM Inequality and (1), we obtain

$$P^2 \leq 9(P+3+x+1+y+2) = 9(2P+6) = 18P+54,$$

or $P^2 - 18P - 54 \leq 0$, which implies that $P \leq 9+3\sqrt{15}$. The proof is complete.

4. Find all real-valued functions f defined on $\mathbb{R}$ that satisfy the identity $f(f(x-y)) = f(x)f(y) - f(x) + f(y) - xy$.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Daniel Tsai, student, Taipei American School, Taipei, Taiwan. We give the write up of Bataille.

It is readily checked that the function $f(x) = -x$ for all x is a solution. We show that there are no other solutions. Let f satisfy

$$f(f(x-y)) = f(x)f(y) - f(x) + f(y) - xy \qquad (1)$$

for all x, $y \in \mathbb{R}$ and let $a = f(0)$. Taking $x = y = 0$ in (1) gives $f(a) = a^2$ and then taking $y = x$ shows that

$$f(x)^2 = x^2 + a^2$$

for all real numbers x. In particular $f(a)^2 = 2a^2$, that is, $a^4 = 2a^2$, hence $a \in \{0, \sqrt{2}, -\sqrt{2}\}$. Assume that $a = \sqrt{2}$. Then for any given x we have $f(x) = \sqrt{x^2+2}$ or $f(x) = -\sqrt{x^2+2}$ and taking $y = 0$ in (1), we obtain

$$f(f(x)) = (\sqrt{2}-1)f(x) + \sqrt{2}\,. \qquad (2)$$

Now, $f(\sqrt{2}\,) = f(a) = a^2 = 2$ so that $f(f(\sqrt{2})) = f(2) = \pm\sqrt{6}$, but then taking $x = \sqrt{2}$ in (2) yields a contradiction since $6 \neq (3\sqrt{2}-2)^2$. Similarly, the assumption $a = -\sqrt{2}$ leads to a contradiction. It follows that $a = 0$, hence $f(x) = x$ or $f(x) = -x$ for any given x. However, if $f(x_0) = x_0$ for some nonzero real number x_0, then $f(f(x_0)) = f(x_0) = x_0$, while from (1) we have $f(f(x_0)) = f(f(x_0 - 0)) = -f(x_0) = -x_0$. This is impossible since $x_0 \neq 0$, hence $f(x) = -x$ for all real x.

5. Find all triples of non-negative integers (x, y, n) such that $\dfrac{x! + y!}{n!} = 3^n$ (with the convention $0! = 1$).

Solved by Daniel Tsai, student, Taipei American School, Taipei, Taiwan; and Konstantine Zelator, University of Toledo, Toledo, OH, USA. We give the solution of Tsai.

Let S be the set of all ordered triples (x, y, n) of nonnegative integers such that $\dfrac{x! + y!}{n!} = 3^n$, or equivalently $x! + y! = 3^n n!$. For $n = 0$, it is seen at once that there is no corresponding $(x, y, n) \in S$. For $n = 1$, let $(x, y, n) \in S$; then if $x \geq 3$ or $y \geq 3$ we have $x! + y! \geq 7 > 3 = 3^1 1!$, thus x, $y < 3$ and simple checking yields that $\{(0, 2, 1), (1, 2, 1), (2, 0, 1), (2, 1, 1)\} \subset S$.

Lemma Let $(x, y, n) \in S$ with $n \geq 2$. Then x, $y \geq n$ and $x > n$ or $y > n$.

Proof: If x, $y \leq n$, then $x! + y! \leq 2n! < 3^n n!$, so $x > n$ or $y > n$. If $x > n$ and $y < n$, then $\dfrac{x! + y!}{n!} = \dfrac{x!}{n!} + \dfrac{y!}{n!} = 3^n$, a contradiction since $\dfrac{x!}{n!}$ is an integer but $\frac{y!}{n!}$ is not. Thus, if $x > n$, then $y \geq n$. Similarly, in the case $y > n$ we have $x \geq n$ by symmetry. ∎

We shall prove that for $n \geq 2$ there is no corresponding $(x, y, n) \in S$ by considering cases on n modulo 3.

Case 1. $n \equiv 0 \pmod 3$. Let $(x, y, n) \in S$ and assume without loss of generality that $x \leq y$. By the Lemma, $n \leq x \leq y$ and one of these two inequalities is strict. If $x > n$, then from $\frac{x! + y!}{n!} = 3^n$ it follows that $(n+1)|3^n$. However, $n+1$ has a prime divisor other than 3, a contradiction. Therefore, $n = x < y$, and consequently

$$\frac{x! + y!}{n!} = \frac{x!}{n!} + \frac{y!}{n!} = 1 + (n+1)(n+2)\cdots y = 3^n .$$

Thus, 3 divides $1 + (n+1)(n+2)\cdots y$ and $(n+1)(n+2)\cdots y \equiv 2 \pmod 3$, which implies that $y = n + 2$. However, $1 + (n+1)(n+2) < 3^n$ for $n \geq 3$ (by induction) and $1 + (2+1)(2+2) \neq 3^2$, contradicting the fact that $(x, y, n) \in S$.

Case 2. $n \equiv 1 \pmod 3$. Let $(x, y, n) \in S$ and assume without loss of generality that $x \leq y$. By reasoning similar to that of Case 1 it follows that $x = n$ and $y = n + 1$. However, $1 + (n+1) < 3^n$ for each integer $n \geq 2$, contradicting the fact that $(x, y, n) \in S$.

Case 3. $n \equiv 2 \pmod 3$. Let $(x, y, n) \in S$ and assume without loss of generality that $x \leq y$. By the Lemma, $n \leq x \leq y$ and one of these two inequalities is strict. If $x \geq n + 2$, then from $\frac{x! + y!}{n!} = 3^n$ it follows that $(n+2)|3^n$. However, $n+2$ has a prime divisor other than 3, a contradiction, hence $n \leq x < n + 2$.

If $x = n$, then $n = x < y$ and

$$\frac{x! + y!}{n!} = \frac{x!}{n!} + \frac{y!}{n!} = 1 + (n+1)(n+2)\cdots y = 3^n ,$$

contradicting $n + 1 \equiv 0 \pmod 3$.

If $x = n + 1$, then

$$\frac{x! + y!}{n!} = \frac{x!}{n!} + \frac{y!}{n!} = (n+1) + (n+1)(n+2)\cdots y = 3^n .$$

If furthermore $y = n + 1$, then

$$2(n+1) = (n+1) + (n+1)(n+2)\cdots y = 3^n$$

is even, a contradiction. Thus, $y > n+1$ and $(n+1)\big(1 + (n+2)\ldots y\big) = 3^n$. It follows that $1 + (n+2)\cdots y \equiv 0 \pmod 3$ and $(n+2)\cdots y \equiv 2 \pmod 3$, which implies that $y = n + 3$. However, $(n+1)\big(1 + (n+2)(n+3)\big) \neq 3^n$

for $1 \leq n \leq 5$ and by induction $(n+1)\big(1+(n+2)(n+3)\big) < 3^n$ for $n \geq 6$, contradicting the fact that $(x, y, n) \in S$.

6. Let the sequence x_1, x_2, x_3, ..., be defined by $x_1 = a$, where a is a real number, and the recursion $x_{n+1} = 3x_n^3 - 7x_n^2 + 5x_n$ for $n \geq 1$.

Find all values of a for which the sequence has a finite limit as n tends to infinity, and find this limit.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Daniel Tsai, student, Taipei American School, Taipei, Taiwan. We give Bataille's write-up.

Let $f(x) = 3x^3 - 7x^2 + 5x$ and $g(x) = f(x) - x = x(x-1)(3x-4)$. The sequence $\{x_n\}$, which satisfies $x_{n+1} = f(x_n)$ for all positive integers n, can only converge to a root of $g(x) = 0$. Thus, the only possible finite limits of $\{x_n\}$ are 0, 1, and $\frac{4}{3}$. We show that the sequence is convergent if and only if $0 \leq a \leq \frac{4}{3}$, in which case the limit is 1 except if $a = 0$ and $\lim\limits_{n\to\infty} x_n = 0$ or if $a = \frac{4}{3}$ and $\lim\limits_{n\to\infty} x_n = \frac{4}{3}$.

Suppose first $a < 0$. Since $g(x) < 0$ when $x < 0$, it follows that $x_n < x_1 = a < 0$ for all positive integers n. If $\{x_n\}$ had a finite limit, ℓ, we would have $\ell \leq a$, contradicting the fact that $\ell \in \{0, 1, \frac{4}{3}\}$. Thus, $\{x_n\}$ is divergent when $a < 0$. Using the fact that $g(x) > 0$ for $x > \frac{4}{3}$, similar reasoning shows that $\{x_n\}$ is divergent when $a > \frac{4}{3}$.

If $a \in \{0, 1, \frac{4}{3}\}$, then the sequence $\{x_n\}$ is constant.

If $a \in (1, \frac{4}{3})$, then using $f(x) - 1 = (x-1)^2(3x-1)$ an easy induction shows that $1 < x_{n+1} < x_n$ for all positive integers n. Thus, $\{x_n\}$ is decreasing and bounded, hence convergent. Its limit ℓ satisfies $\ell \geq 1$ and $\ell \in \{0, 1, \frac{4}{3}\}$, that is, $\ell = 1$.

If $a \in [\frac{1}{3}, 1)$ then $x_2 = f(a) \geq 1$ and $x_2 < \frac{4}{3}$, as the maximum of f on $[0, 1]$ is $f\left(\frac{5}{9}\right) = \frac{275}{243} < \frac{4}{3}$. From the previous case, we see that $\lim\limits_{n\to\infty} x_n = 1$.

It remains to study the case $a \in \left(0, \frac{1}{3}\right)$. Then, $\dfrac{1}{3^{m+1}} \leq a < \dfrac{1}{3^m}$ for some unique positive integer m. If any of the numbers x_2, x_3, ..., x_m is not less than $\frac{1}{3}$, let x_k be the one with the smallest index. Then $\frac{1}{3} \leq x_k < \frac{4}{3}$ and by the previous cases $\{x_n\}_{n\geq k}$ converges to 1 and $\lim\limits_{n\to\infty} x_n = 1$. Otherwise, noting that $f(x) - 3x = x(x-2)(3x-1)$ is positive for $x \in \left(0, \frac{1}{3}\right)$, we have

$$\begin{aligned} x_2 &= f(x_1) > 3x_1 = 3a \geq \frac{1}{3^m}, \\ x_3 &= f(x_2) > 3x_2 \geq \frac{1}{3^{m-1}}, \\ &\cdots \\ x_m &= f(x_{m-1}) > 3x_{m-1} \geq \frac{1}{3^2}, \end{aligned}$$

and finally $x_{m+1} > \frac{1}{3}$. So $\{x_n\}_{n\geq m+1}$ converges to 1 and again $\lim\limits_{n\to\infty} x_n = 1$.

To finish this number of the *Corner* we give solutions from the readers to problems of the 2005 German Mathematical Olympiad, given at [2008 : 82].

1. Determine all pairs (x, y) of reals, which satisfy the system of equations

$$\begin{aligned} x^3 + 1 - xy^2 - y^2 &= 0\,, \\ y^3 + 1 - x^2y - x^2 &= 0\,. \end{aligned}$$

Solved by George Apostolopoulos, Messolonghi, Greece; Konstantine Zelator, University of Toledo, Toledo, OH, USA; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Apostolopoulos.

We subtract the two equations of the system to obtain

$$\left(x^3 - y^3\right) \;+\; xy(x-y) \;+\; x^2 - y^2 \;=\; 0\,,$$

which upon factoring becomes

$$(x-y)(x+y)(x+y+1) \;=\; 0\,.$$

Thus, $y = x$ or $y = -x$ or $y = -x - 1$.

If $y = x$, then $x = \pm 1$, hence $(x, y) = (1, 1)$ or $(x, y) = (-1, -1)$.
If $y = -x$, then again $x = \pm 1$, hence $(x, y) = (1, -1)$ or $(x, y) = (-1, 1)$.
If $y = -x - 1$ we substitute into the first equation to obtain

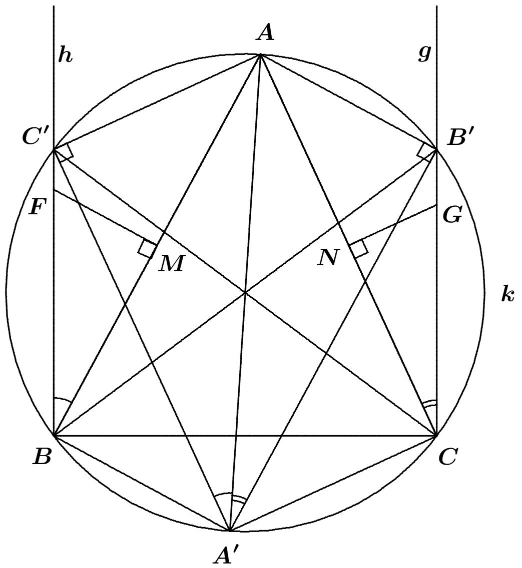
$$x^3 + 1 - x(x+1)^2 - (x+1)^2 \;=\; -3x(x+1) \;=\; 0\,,$$

hence $x = 0$ or $x = -1$ and $(x, y) = (0, -1)$ or $(x, y) = (-1, 0)$.

2. Let A, B, and C be three distinct points on the circle k. Let the lines h and g each be perpendicular to BC with h passing through B and g passing through C. The perpendicular bisector of AB meets h in F and the perpendicular bisector of AC meets g in G. Prove that the product $|BF| \cdot |CG|$ is independent of the choice of A, whenever B and C are fixed.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Konstantine Zelator, University of Toledo, Toledo, OH, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Amengual Covas.

Let A' be the point diametrically opposite to A. Let M and N be the midpoints of the segments AB and AC, respectively. Let B' and C' be the second points of intersection of the lines g and h with the circle k, respectively.



Since $C'BC$ and BCB' are right angles, the segments BB' and CC' are diameters of the circle k. Thus, the two quadrilaterals $AC'A'C$ and $ABA'B'$ are rectangles, and hence parallelograms, so we have

$$|C'A'| = |CA| \quad \text{and} \quad |A'B'| = |AB|\,.$$

Since the right triangles BMF and $A'C'A$ are similar, as are the right triangles CNG and $A'B'A$, we have

$$\frac{BF}{AA'} = \frac{BM}{C'A'} = \frac{\frac{1}{2}AB}{CA}$$

and

$$\frac{CG}{AA'} = \frac{CN}{A'B'} = \frac{\frac{1}{2}CA}{AB}\,,$$

from which we obtain

$$BF \cdot CG = \frac{1}{4}|AA'|^2$$

as the square of the radius of k, which is independent of the choice of A.

3. A lamp is placed at each lattice point (x, y) in the plane (that is, x and y are both integers). At time $t = 0$ exactly one lamp is switched on. At any integer time $t \geq 1$, exactly those lamps are switched on which are at a distance of 2005 from some lamp which is already switched on. Prove that every lamp will be switched on at some time.

Solution by Titu Zvonaru, Comănești, Romania.

Assume that at time $t = 0$ the lamp at $O(0, 0)$ is switched on. Since $2005 = \sqrt{1357^2 + 1476^2}$ then at some time the lamps at the following lattice points will be switched on:

$$\begin{array}{rl} A_1(1357, 1476)\,, & O_1(2\cdot 1357, 0)\,, \\ A_2(3\cdot 1357, 1476)\,, & O_2(4\cdot 1357, 0)\,, \\ \vdots & \vdots \\ A_k\big((2k-1)\cdot 1357, 1476\big)\,, & O_k(2k\cdot 1357, 0)\,; \end{array}$$

and then the lamps at these lattice points will be switched on:

$$\begin{array}{l} B_1(2k\cdot 1357 - 2005, 0)\,, \\ B_2(2k\cdot 1357 - 2\cdot 2005, 0)\,, \\ B_3(2k\cdot 1357 - 3\cdot 2005, 0)\,, \\ \vdots \\ B_t(2k\cdot 1357 - t\cdot 2005, 0)\,. \end{array}$$

The equation $2k\cdot 1357 - 2005t = 1$ is the same as $2714k - 2005t = 1$, which has a solution in positive integers k, t because $\gcd(2714, 2005) = 1$, for example, $2714\cdot 1134 - 2005\cdot 1535 = 1$. Thus the lamp at $(1, 0)$ will be switched on at some time. It follows (by symmetry) that every lamp will be switched on at some time.

4. Let $Q(n)$ denote the sum of the digits of the positive integer n. Prove that $Q\Big(Q\big(Q(2005^{2005})\big)\Big) = 7$.

Solution by Titu Zvonaru, Comănești, Romania.

It is well known that $Q(n) \equiv n \pmod 9$. Let $k = Q\Big(Q\big(Q(2005^{2005})\big)\Big)$, then $k \equiv 2005^{2005} \pmod 9$ and we have

$$\begin{aligned} 2005^{2005} &\equiv (-2)^{2005} = -2\cdot 2^{2004} = -2\cdot\left(2^3\right)^{668} \\ &\equiv -2(-1)^{668} = -2 \equiv 7 \pmod 9\,, \end{aligned}$$

so that $k \equiv 7 \pmod 9$.

The number 2005^{2005} has at most $4\cdot 2005 = 8020$ digits. Hence, $Q(2005^{2005})$ is at most $9\cdot 8020 = 72180$. This implies that

$$Q\Big(Q\left(2005^{2005}\right)\Big) \le 5\cdot 9 = 45$$

and hence $k = Q\Big(Q\big(Q(2005^{2005})\big)\Big)$ is at most $Q(39) = 12$.

Altogether, k is an integer satisfying $0 \le k \le 12$ and $k \equiv 7 \pmod 9$, hence $k = 7$, as desired.

That completes the *Corner* for this issue. Send me your nice solutions, generalizations, and Olympiad problem sets.

BOOK REVIEWS

Amar Sodhi

Putnam and Beyond
By Răzvan Gelca and Titu Andreescu, Springer Science + Business Media LLC, New York, 2007
ISBN-13: 978-0-387-25765-5, softcover, 798+xvi pages, US$69.95
e-ISBN-13: 978-0-387-68445-1
Reviewed by **Jeff Hooper**, *Acadia University, Wolfville, NS*

One of my favourite problem books is the one by Andreescu and Gelca, *Mathematical Olympiad Challenges* (Birkhäuser, 2000), in which the authors collect numerous problems centred mainly around past Olympiads, and group them together by similar topic. In *Putnam and Beyond*, the authors again combine to deliver a similar problem book, based this time around on the Putnam Competition.

Structurally, the book consists of six major chapters: Methods of Proof, Algebra, Real Analysis, Geometry and Trigonometry, Number Theory, and Combinatorics and Probability. Each chapter is divided into numerous sections and subsections, each of which focuses on an important problem idea. For instance, the initial section of the Algebra chapter is *Identities and Inequalities*, and here we find the topic divided into a number of important ideas: Algebraic Identities, the equation $x^2 \geq 0$, the Cauchy-Schwartz Inequality, the Triangle Inequality, the AM–GM Inequality, and so on. Each section contains explanations of the key ideas and several worked problems, along with a number of problems to solve. Full solutions are provided at the back of the book, along with sources and/or helpful references where necessary. The initial chapter is a particularly nice introduction for students.

There is some overlap, of course, with the topics covered in the authors' previous book, and I was expecting there to be much in common. To their credit, the authors have avoided that sort of mild cheating: for the topics in common with their first book, the examples and problems they offer are new. This new book is also a much larger and far more extensive effort. In addition to all of the examples provided, the book contains more than 900 problems. And these are Putnam-level problems, so they are mainly a level up from the earlier book. In fact the topics in the book go very deep, and cover most of the major ideas found in the undergraduate mathematics curriculum. So there's lots here.

The book is not error-free, however, and a teacher or coach who uses this book should be a little careful. I certainly haven't worked through every problem in the book, but did dip into them in a number of sections. Just to give an example, Problem 33 asks the solver, "Given 50 distinct positive integers strictly less than 100, prove that some two of them sum to 99." A little thought shows that one can take the numbers 50, 51, $\ldots$, 99 and the statement fails.

But these problems are of the minor variety. This wonderful book is an excellent problems resource and should become a part of any serious library for problem solving. By collecting together problems by topic, the authors provide readers the chance to study each of these important problem-solving techniques and ideas in isolation, and help them begin to see the inherent patterns. This should be one of the first books considered as a resource by anyone coaching groups of problem solvers.

Mathematical Connections: A Companion for Teachers and Others
By Al Cuoco, Mathematical Association of America, 2005
ISBN 978-0-88385-739-7, hardcover, 239+xix pages, US$54.95
Reviewed by **Peter S. Brouwer**, *State University of New York, Potsdam, NY, USA*

Al Cuoco is the Director of the Center for Mathematics Education at the Education Development Center in Newton, MA, where he works in the areas of curriculum development and professional development of teachers. This book joins a number of recent others in addressing secondary school mathematics content topics from an advanced (or deeper) perspective. The primary audience is high school mathematics teachers, and the book is based on the assumption that providing a more advanced treatment of some of the mathematical topics taught at that level is a valuable form of professional development. The author's emphasis is on making connections between topics and developing mathematical habits of mind.

The choice of topics is somewhat idiosyncratic, and reflects the interrelated topics that Cuoco is interested in exploring. The chapter titles are: 1. Difference Tables and Polynomial Fits, 2. Form and Function: The Algebra of Polynomials, 3. Complex Numbers, Complex Maps, and Trigonometry, 4. Combinations and Locks, and 5. Sums of Powers.

The strength of this book is that it is essentially a problems book (on the above topics). There are many problems, including 90 in the first chapter alone, and the reader is asked to work them sequentially while reading through the text. These are grouped by themes, which aids coherence. In addition, there are many problems given as exercises. The author includes helpful notes on selected problems at the end of each chapter.

Many of the problems in this book are quite challenging, but its incremental and thematic approach helps. As polynomial algebra (and patterns in polynomial coefficients) appear in every chapter, the reader must be comfortable manipulating rather complicated algebraic expressions. I would recommend this book for serious, mathematics-based professional development programs as well as for experienced independent readers who would enjoy pursuing a fruitful intellectual journey through selected advanced secondary mathematics topics.

Velocity Analysis: an Approach to Solving Geometry Problems

Peng YuChen

1 Introduction

We introduce velocity analysis for solving otherwise complex geometry problems, then we give two examples of the use of this method: (1) to give a very brief proof of the optical property of the ellipse, and (2) to find the length of a logarithmic spiral. At the end of this note we leave some problems for the reader; the solutions will be very brief if velocity analysis is used.

2 Velocity Analysis

In analytic geometry, a curve is a set of points whose coordinates (x, y) satisfy a certain formula. For example, the set of points $\{(x, y) : x^2 + y^2 = R^2\}$ is a circle or radius R.

As we all know, the trace of a moving point is a curve. In the analytic notion of a curve above, we mainly pay attention to the *position* of a moving point, rather than its velocity. In fact, either the position or the velocity of a moving point can describe the course of its motion, and in some cases it is more convenient to study the velocity rather than the position. If instead of studying the coordinates of a moving point we study its velocity, then it is very simple for us to deduce certain geometric properties of the traced curve without engaging in complex mathematics, especially when seeking the tangent or arc length. (As we know, the velocity vector of a moving point gives a tangent to the curve traced by the point.)

Now we introduce the basic idea of velocity analysis in solving geometry problems.

A point P is moving under a certain restriction on its velocity $\overrightarrow{V}$. For example, if $\overrightarrow{V} \cdot \overrightarrow{OP} \equiv 0$ (here $\overrightarrow{OP}$ is the position vector of P), then obviously the point P traces out a circle whose centre is O. That is to say, we find an equivalent way of defining the circle by studying the velocity rather than the coordinates of a moving point.

When studying velocity, usually it is more convenient to study components. In our example of the circle, we break the velocity of P into two directions: along $\overrightarrow{OP}$ and orthogonal to $\overrightarrow{OP}$. Name the two components $\overrightarrow{V_1}$

and $\overrightarrow{V_2}$ respectively; if the component $\overrightarrow{V_1}$ along $\overrightarrow{OP}$ satisfies $\overrightarrow{V_1} \equiv 0$, then the point P traces out a circle.

This is the basic idea of velocity analysis: if we decompose the velocity of a moving point into two appropriate directions, giving certain restrictions to the components of the velocity, the trace of the moving point will become a certain curve. In this setting it is very easy to analyse the tangent to the curve, for we already know the restriction on the velocity of the point.

Example 1 The optical property of the ellipse. To show the power of the vector analysis approach, we use it to solve this classical geometry problem. In ancient times people found that conic sections have very special and beautiful optical properties. One example is this: if a ray of light leaves one focus of an ellipse and strikes the ellipse, it will be reflected to the other focus of the ellipse (see Figure 1).

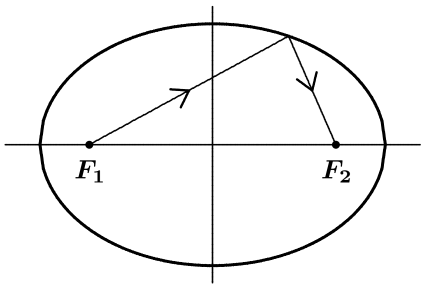


Figure 1

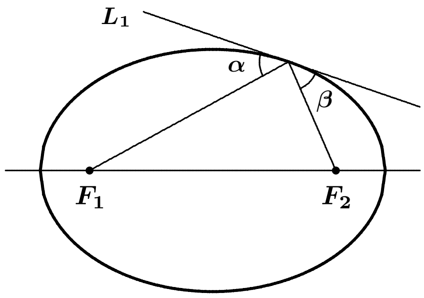


Figure 2

Using calculus to prove this property would be complicated, so we use the approach of analysing the velocity of a moving point to give a much shorter proof. As in Figure 2, to prove the optical property of the ellipse, we need to prove that $\alpha = \beta$, which is enough to satisfy the Law of Reflection. Here L_1 is tangent to the ellipse.

We usually define an ellipse like this: two points F_1 and F_2 in the plane are fixed. A point P moves so that $|\overrightarrow{PF_1}|+|\overrightarrow{PF_2}|$ is always a constant. We call the trace of P an ellipse. In accordance with the idea of velocity analysis, we describe an ellipse as follows:

Definition. Two points F_1, F_2 in the plane are fixed (see Figure 3). A moving point, P, has velocity vector $\overrightarrow{V}$. Let $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ be the components of $\overrightarrow{V}$ towards F_1 and away from F_2, respectively. If $|\overrightarrow{V_1}| \equiv |\overrightarrow{V_2}|$, then the trace of P is an ellipse.

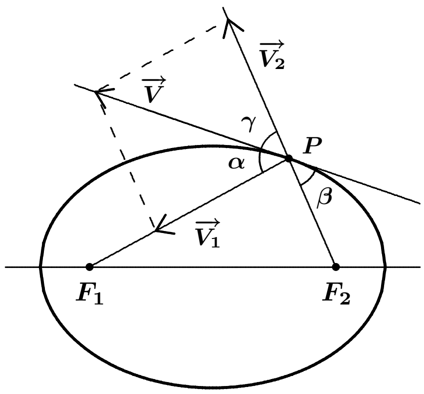


Figure 3

It is obvious that the two definitions are equivalent. For when P moves towards F_1 a short distance, it must then move away from F_2 by that same distance to ensure that $|\overrightarrow{PF_1}| + |\overrightarrow{PF_2}|$ is always a constant.

Now $\overrightarrow{V}$ is a tangent vector of the ellipse. Since $|\overrightarrow{V_1}| \equiv |\overrightarrow{V_2}|$, then $\alpha = \gamma$. Since $\beta = \gamma$, we also have $\alpha = \beta$. That is all.

Example 2 The length of the logarithmic spiral. As another example, we find the length of the logarithmic spiral $\rho = \rho_0 e^{a\theta}$ between $\rho = \rho_1$ and $\rho = \rho_2$. Usually we cannot figure out this problem without using a lot of calculus, so we introduce a physical model.

As in Figure 4, three points A, B, and C are each at a vertex of an equilateral triangle of side $\sqrt{3}\rho_0$. Point A moves towards B, B moves towards C, and C moves towards A. The speed of each point is s. Where will the three points meet and how far will they travel before meeting?

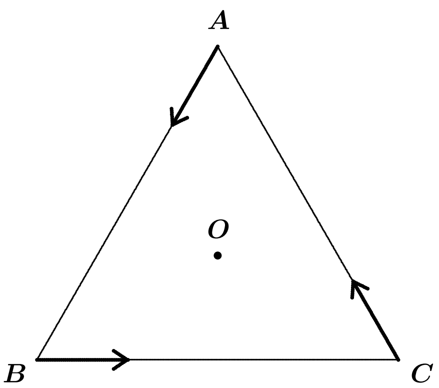


Figure 4

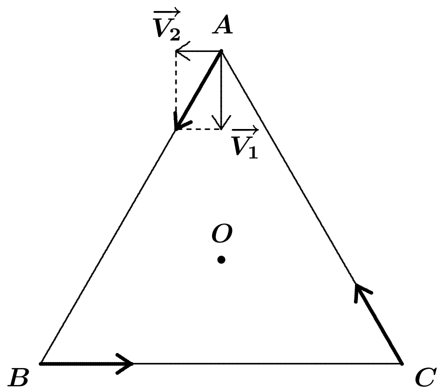


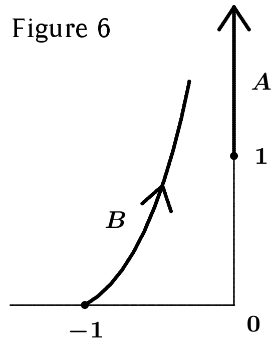
Figure 5

The solution of this problem is easy: obviously the three points will meet at the point O, which is the centre of the triangle. We decompose the velocity $\overrightarrow{V}$ of A at each instant into two components: one towards the point O and the other perpendicular to $\overrightarrow{AO}$, and we call these two components $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$, respectively. We have that $|\overrightarrow{V_1}| = |\overrightarrow{V}|\cos 30° = \frac{\sqrt{3}}{2}s$ (see Figure 5), so the time until they meet is $t = \frac{\rho_0}{|\overrightarrow{V_1}|} = \frac{\rho_0}{(\sqrt{3}/2)s}$ and the distance traveled is $d = st = \frac{2\rho_0}{\sqrt{3}}$.

Now we consider the curve traced by A. We use a polar coordinate system with the origin at O and with the vector $\overrightarrow{V_2}$ pointing in the direction of increasing θ. At each instant, $|\overrightarrow{V_2}| = |\overrightarrow{V}|\sin 30° = |\overrightarrow{V_1}|\tan 30°$. Since $|\overrightarrow{V_2}| = \frac{\rho d\theta}{dt}$ and $|\overrightarrow{V_1}| = -\frac{d\rho}{dt}$, we have $d\rho = a\rho d\theta$, the differential equation of the spiral $\rho = \rho_0 e^{a\theta}$ with $a = \cot 150° = -\sqrt{3}$. Thus, the length of the curve from $\rho = \rho_1$ to $\rho = \rho_2$ is $\frac{|\rho_1 - \rho_2|}{|\overrightarrow{V_1}|} \cdot s = \frac{2}{\sqrt{3}}|\rho_1 - \rho_2|$. We leave the problem of finding a suitable physical model for other values of a to the reader.

We leave three more problems for the reader to solve. They could be solved by using analytic geometry and calculus, but are more conveniently solved using the approach of velocity analysis.

Figure 6



1. The pursuant trajectory problem. As in Figure 6, suppose that an object A starts from the point $(0, 1)$, and moves with a constant speed s in the direction of the positive y–axis. At the same time another object B starts from the point $(-1, 0)$, and moves with speed $2s$ and always in the direction of the object A. When will the objects meet?

2. The optical property of a hyperbola. If a light ray leaves one focus of a hyperbola and strikes the hyperbola, then the (reverse) extended line of its reflection will pass through the other focus of the hyperbola (see Figure 7).

3. The optical property of a parabola. If a light ray leaving the focus is reflected in the parabola, then its reflection is parallel to the axis of symmetry (see Figure 8).

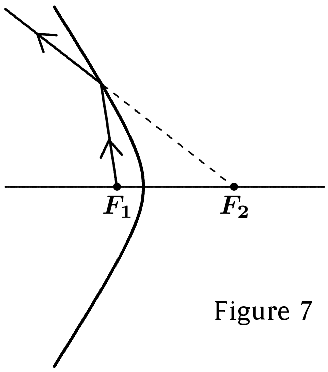


Figure 7

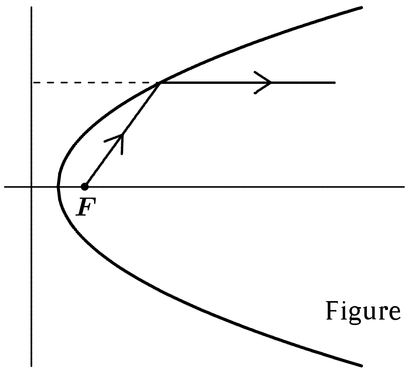


Figure 8

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 August 2009**. *An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet of the University College of Saint Boniface and Jean-Marc Terrier of the University of Montreal for translations of the problems.

3401. *Proposed by Tigran Sloyan, Basic Gymnasium of SEUA, Yerevan, Armenia.*

Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle EAD$ and $\angle BCA = \angle EDA$, and let the lines CB and DE intersect in the point F. Prove that the midpoints of CD, BE, and AF are collinear.

3402. *Proposed by Mihály Bencze, Brasov, Romania.*

Let D and E be the midpoints of the sides AB and AC in triangle ABC, respectively. Prove that CD is perpendicular to BE if and only if

$$5BC^2 = AC^2 + AB^2 .$$

3403. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

The circles Γ_1 and Γ_2 intersect at P and Q. A line ℓ through P intersects Γ_1 and Γ_2 for the second time at A and B, respectively. The tangents to Γ_1 and Γ_2 at A and B intersect at C. If O is the circumcentre of $\triangle ABC$ determine the locus of O when ℓ rotates about P.

3404. *Proposed by Michel Bataille, Rouen, France.*

Let Q be a cyclic quadrilateral. The perpendiculars to each diagonal through its endpoints form a parallelogram, P. Characterize the centre of P and show that opposite sides of Q intersect on a diagonal of P.

3405. *Proposed by Michel Bataille, Rouen, France.*

Find the minimum value of

$$|\cos\alpha| + |\cos\beta| + |\cos\gamma| + |\cos(\alpha-\beta)| + |\cos(\beta-\gamma)| + |\cos(\gamma-\alpha)| ,$$

where α, β, and γ are real numbers.

3406. *Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find
$$\lim_{n\to\infty} \ln\left[\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right] .$$

3407. *Proposed by Roy Barbara, Lebanese University, Fanar, Lebanon.*

Let S be a set of positive integers containing the integer 2007 and such that

(a) If x, $y \in S$ and $x \neq y$, then $|x-y| \in S$, and

(b) If $x \in S$, then $\left(x^3 - 1007x + 3007\right) \in S$.

Prove that S is the set of all positive integers.

3408. *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.*

Let $\{c_i\}_{i=1}^{\infty}$ be a sequence of distinct positive integers, and let $|q| < 1$. Prove that the inequality
$$\frac{\displaystyle\sum_{i=1}^{\infty} c_i q^{c_i}}{1+\displaystyle\sum_{i=1}^{\infty} q^{c_i}} \leq \frac{q}{1-q}$$
holds for all such sequences $\{c_i\}_{i=1}^{\infty}$ if and only if $q \in \left[0, \frac{1}{2}\right]$.

3409. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c, and d be positive real numbers. Prove that
$$\frac{ab+bc+ca}{a^3+b^3+c^3} + \frac{ab+bd+da}{a^3+b^3+d^3} + \frac{ac+cd+da}{a^3+c^3+d^3} + \frac{bc+cd+db}{b^3+c^3+d^3}$$
$$\leq \min\left\{ \frac{a^2+b^2}{(ab)^{3/2}} + \frac{c^2+d^2}{(cd)^{3/2}}, \frac{a^2+c^2}{(ac)^{3/2}} + \frac{b^2+d^2}{(bd)^{3/2}}, \frac{a^2+d^2}{(ad)^{3/2}} + \frac{b^2+c^2}{(bc)^{3/2}} \right\}.$$

3410. *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, and c be the sides of triangle ABC, let R be its circumradius, and let F be its area. Prove that
$$\sum_{\text{cyclic}} \frac{bc\sin^2 A/2}{b+c} \geq \frac{F}{2R} .$$

3411. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers such that

$$a^6 + b^6 + c^6 \;<\; \frac{32}{33}\left(a^3 + b^3 + c^3\right)^2 \,.$$

Prove that at least one of the quadratics $ax^2 + bx + c$, $bx^2 + cx + a$, or $cx^2 + ax + b$ has no real roots.

3412. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$\sum_{\text{cyclic}} \frac{1}{\sqrt{a^3 + 2b^3 + 6}} \;\leq\; 1 \,.$$

3413. *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, c, and d be real numbers in the interval $[1, 2]$. Prove that

$$\frac{a+b}{c+d} + \frac{c+d}{a+b} - \frac{a+c}{b+d} \;\leq\; \frac{3}{2} \,.$$

. . . .

3401. *Proposé par Tigran Sloyan, Lycée de la SEUA, Erevan, Arménie.*

Soit $ABCDE$ un pentagone convexe tel que $\angle BAC = \angle EAD$ et $\angle BCA = \angle EDA$. Soit F le point d'intersection des droites CB et DE. Montrer que les milieux des CD, BE et AF sont colinéaires.

3402. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit le triangle ABC, où D et E sont les milieux des côtés AB et AC, respectivement. Montrer que CD est perpendiculaire à BE si et seulement si

$$5BC^2 \;=\; AC^2 + AB^2 \,.$$

3403. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Les cercles Γ_1 et Γ_2 intersectent à P et Q. Une ligne ℓ passant par P intersecte une seconde fois Γ_1 et Γ_2 à A et B, respectivement. Les tangentes de Γ_1 et Γ_2 à A et B intersectent à C. Si O est le centre du cercle circonscrit du $\triangle ABC$ dèterminer le lieu de O lorsque ℓ tourne autour de P.

3404. *Proposé par Michel Bataille, Rouen, France.*

Soit Q un quadrilatère cyclique. Les perpendiculaires à chaque diagonale issues de ses sommets forment un parallèlogramme P. Caractériser le centre de P et montrer que les côtés opposés de Q se coupent sur une diagonale de P.

3405. *Proposé par Michel Bataille, Rouen, France.*

Déterminer la valeur minimum de

$$|\cos\alpha| + |\cos\beta| + |\cos\gamma| + |\cos(\alpha-\beta)| + |\cos(\beta-\gamma)| + |\cos(\gamma-\alpha)|\,,$$

où α, β et γ sont des nombres réels.

3406. *Proposé par José Luis Díaz-Barrero et Miquel Grau-Sánchez, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Calculer

$$\lim_{n\to\infty} \ln\left[\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right]\,.$$

3407. *Proposé par Roy Barbara, Université Libanaise, Fanar, Liban.*

Soit S un ensemble d'entiers contenant l'entier 2007 et tel que

(a) Si x, $y \in S$ et $x \neq y$, alors $|x-y| \in S$, et

(b) Si $x \in S$, alors $\left(x^3 - 1007x + 3007\right) \in S$.

Montrer que S est l'ensemble de tous les entiers positifs.

3408. *Proposé par Slavko Simic, Institut de Mathématiques SANU, Belgrade, Serbie.*

Soit $\{c_i\}_{i=1}^{\infty}$ une suite d'entiers positifs distincts, et soit $|q| < 1$. Montrer que l'inégalité

$$\frac{\displaystyle\sum_{i=1}^{\infty} c_i q^{c_i}}{\displaystyle 1+\sum_{i=1}^{\infty} q^{c_i}} \leq \frac{q}{1-q}$$

vaut pour toutes les suites de ce type si et seulement si $q \in \left[0, \frac{1}{2}\right]$.

3409. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b, c et d des nombres réels positifs. Montrer que

$$\frac{ab+bc+ca}{a^3+b^3+c^3} + \frac{ab+bd+da}{a^3+b^3+d^3} + \frac{ac+cd+da}{a^3+c^3+d^3} + \frac{bc+cd+db}{b^3+c^3+d^3}$$
$$\leq \min\left\{ \frac{a^2+b^2}{(ab)^{3/2}} + \frac{c^2+d^2}{(cd)^{3/2}},\ \frac{a^2+c^2}{(ac)^{3/2}} + \frac{b^2+d^2}{(bd)^{3/2}},\ \frac{a^2+d^2}{(ad)^{3/2}} + \frac{b^2+c^2}{(bc)^{3/2}} \right\}.$$

3410. *Proposé par Joe Howard, Portales, NM, É-U.*

Soit a, b et c les côtés du triangle ABC, soit R le rayon de son cercle circonscrit et F son aire. Montrer que

$$\sum_{\text{cyclique}} \frac{bc\sin^2 A/2}{b+c} \;\geq\; \frac{F}{2R}\,.$$

3411. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c des nombres réels positifs tels que

$$a^6+b^6+c^6 \;<\; \frac{32}{33}\left(a^3+b^3+c^3\right)^2\,.$$

Montrer que au moins une des quadratiques ax^2+bx+c, bx^2+cx+a et cx^2+ax+b n'a aucune racine réelle.

3412. *Proposé par Cao Minh Quang, Collège Nguyen Binh Khiem, Vinh Long, Vietnam.*

Soit a, b et c trois nombres réels positifs tels que $abc=1$. Montrer que

$$\sum_{\text{cyclique}} \frac{1}{a^3+2b^3+6} \;\leq\; 1\,.$$

3413. *Proposé par Vo Quoc Ba Can, Université de Médecine et Pharmacie de Can Tho, Can Tho, Vietnam.*

Soit a, b, c et d quatre nombres réels dans l'intervalle $[1,2]$. Montrer que

$$\frac{a+b}{c+d} \;+\; \frac{c+d}{a+b} \;-\; \frac{a+c}{b+d} \;\leq\; \frac{3}{2}\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Last year we received a batch of correct solutions from Steven Karp, student, University of Waterloo, Waterloo, ON, to problems 3289, 3292, 3294, 3296, 3297, 3298, and 3300, which did not make it into the December issue due to being misfiled. Our apologies for this oversight.

3301. [2008 : 44, 46] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Prove that

$$\sum_{n=1}^{\infty} \frac{\ln 2 - \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)}{n} = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{(2n+1)(2n+2)}\,.$$

What is this common value?

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain, expanded by the editor.

Let A and B denote the summations on the left side and the right side of the proposed equality, respectively. Also, let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then

$$\begin{aligned} \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &= H_{2n} - H_n \\ &= H_{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) = \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j}\,. \end{aligned}$$

Since it is well known that $\sum\limits_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} = \ln 2$, by changing the order of the double summation we have

$$\begin{aligned} A &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{j=2n+1}^{\infty} \frac{(-1)^{j-1}}{j} \right) = \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} \left(\sum_{n=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{1}{n} \right) \\ &= \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} H_{\lfloor \frac{j-1}{2} \rfloor} = \sum_{k=1}^{\infty} H_k \left(\frac{1}{2k+1} - \frac{1}{2k+2} \right) \\ &= \sum_{k=1}^{\infty} \frac{H_k}{(2k+1)(2k+2)} = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{(2n+1)(2n+2)} = B\,. \end{aligned}$$

To find the common value of the two absolutely convergent series, let

$$f(x) \;=\; \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} H_{\lfloor \frac{j-1}{2} \rfloor} x^j \,,$$

where the power series for f converges for all $x \in (-1,1)$. Then

$$\begin{aligned} f'(x) \;&=\; \sum_{j=3}^{\infty} (-1)^{j-1} H_{\lfloor \frac{j-1}{2} \rfloor} x^{j-1} \;=\; \sum_{j=2}^{\infty} (-1)^{j} H_{\lfloor \frac{j}{2} \rfloor} x^{j} \\ &=\; \sum_{n=1}^{\infty} \left(H_n x^{2n} - H_n x^{2n+1} \right) \;=\; (1-x) \sum_{n=1}^{\infty} H_n x^{2n} \,. \end{aligned} \tag{1}$$

Now, it is well known that

$$\frac{1}{1-x} \ln\left(\frac{1}{1-x}\right) \;=\; \sum_{n=1}^{\infty} H_n x^n \,. \tag{2}$$

[*Ed*: Multiply $\frac{1}{1-x} = 1+x+x^2+\cdots$ with $-\ln(1-x) = x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots$ and observe that the coefficient of x^n is $1+\frac{1}{2}+\cdots+\frac{1}{n}$.]

From (2) we have $(1-x)\sum\limits_{n=1}^{\infty} H_n x^n = -\ln(1-x)$.

Thus, $(1-x^2)\sum\limits_{n=1}^{\infty} H_n x^{2n} = -\ln(1-x^2)$, and it follows that

$$(1-x)\sum_{n=1}^{\infty} H_n x^{2n} \;=\; -\frac{1}{1+x} \ln(1-x^2) \,. \tag{3}$$

From (1) and (3) we obtain

$$f'(x) \;=\; -\frac{1}{1+x} \ln(1-x^2) \,.$$

Since $f(0) = 0$ and the last improper integral below is convergent, by applying Abel's Continuity Theorem for power series we have

$$A \;=\; \lim_{x \to 1^-} f(x) \;=\; \int_0^1 f'(x)\,dx \;=\; -\int_0^1 \frac{\ln\left(1-x^2\right)}{1+x}\,dx \,. \tag{4}$$

It remains to evaluate the last integral in (4).

With the change of variable $x = 2u-1$, we have

$$\int_0^1 \frac{\ln\left(1-x^2\right)}{1+x}\,dx \;=\; \int_{1/2}^1 \frac{\ln\bigl(4u(1-u)\bigr)}{u}\,du \;=\; I_1 + I_2 \,, \tag{5}$$

where

$$I_1 = \int_{1/2}^{1} \frac{\ln(4u)}{u}\,du = \frac{1}{2}\ln^2(4u)\Big|_{1/2}^{1} = \frac{1}{2}\left((\ln 4)^2 - (\ln 2)^2\right)$$
$$= \frac{1}{2}(\ln 4 + \ln 2)(\ln 4 - \ln 2) = \frac{1}{2}(\ln 8)(\ln 2) = \frac{3}{2}(\ln 2)^2 . \quad (6)$$

On the other hand, using integration by parts and then making the change of variable $u = 1 - t$, we have

$$I_2 = \int_{1/2}^{1} \frac{\ln(1-u)}{u}\,du = (\ln u)\big(\ln(1-u)\big)\Big|_{1/2}^{1} + \int_{1/2}^{1} \frac{\ln u}{1-u}\,du$$
$$= -(\ln 2)^2 + \int_0^{1/2} \frac{\ln(1-t)}{t}\,dt = -(\ln 2)^2 + \int_0^1 \frac{\ln(1-t)}{t}\,dt - I_2 ,$$

from which we obtain

$$2I_2 = -(\ln 2)^2 + \int_0^1 \frac{\ln(1-t)}{t}\,dt . \quad (7)$$

[Ed : Since the integrals in the computations above are improper, care must be taken ; e.g., the evaluation of $(\ln u)\big(\ln(1-u)\big)$ at $u = 1$ must be done by computing $\lim\limits_{u\to 1^-} (\ln u)\big(\ln(1-u)\big)$ using L'Hôspital's Rule.]

Finally,

$$\int_0^1 \frac{\ln(1-t)}{t}\,dt = -\int_0^1 \frac{1}{t}\ln\left(\frac{1}{1-t}\right)dt$$
$$= -\int_0^1 \frac{1}{t}\left(\sum_{n=1}^{\infty} \frac{t^n}{n}\right)dt = -\sum_{n=1}^{\infty}\int_0^1 \frac{t^{n-1}}{n}\,dt$$
$$= -\sum_{n=1}^{\infty}\left(\frac{t^n}{n^2}\Big|_0^1\right) = -\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{\pi^2}{6} . \quad (8)$$

From (4) – (8), we conclude that $A = B = \dfrac{\pi^2}{12} - (\ln 2)^2$.

Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. There were also three incomplete solutions, all of which only demonstrated that the two given summations are equal.

3302. [2008 : 44, 47] *Proposed by Mihály Bencze, Brasov, Romania.*

Let s, r, and R denote the semiperimeter, the inradius, and the circumradius of a triangle ABC, respectively. Show that

$$(s^2 + r^2 + 4Rr)(s^2 + r^2 + 2Rr) \geq 4Rr(5s^2 + r^2 + 4Rr) ,$$

and determine when equality holds.

Solution by Michel Bataille, Rouen, France.

First, using the two known formulas $s^2+r^2+4Rr=ab+bc+ca$ and $abc=4Rrs$, where a, b, and c are the sides of the triangle, we deduce that

$$\begin{aligned}(a+b)(b+c)(c+a) &= (a+b+c)(ab+bc+ca)-abc\\ &= 2s\left(s^2+r^2+4Rr\right)-4Rrs\\ &= 2s\left(s^2+r^2+2Rr\right)\,.\end{aligned}$$

For convenience, let $e_1=a+b+c$, $e_2=ab+bc+ca$, and $e_3=abc$. It follows that the required inequality is successively equivalent to

$$\begin{aligned}(a+b)(b+c)(c+a)e_2 &\geq 8Rrs(5s^2+r^2+4Rr)\,,\\ (a+b)(b+c)(c+a)e_2 &\geq 2e_3\left(e_1^2+e_2\right),\\ \left(ab^2+a^2b+bc^2+b^2c+ca^2+c^2a\right)e_2 &\geq 2e_3e_1^2\,,\\ a^2b^3+a^3b^2+b^2c^3+b^3c^2+c^2a^3+c^3a^2 &\geq 2a^2b^2c+2a^2bc^2+2ab^2c^2\,,\end{aligned}$$

and finally to

$$a^2(b-c)^2(b+c)+b^2(c-a)^2(c+a)+c^2(a-b)^2(a+b)\ \geq\ 0\,.$$

The last inequality is obviously true, which completes the proof. Equality holds if and only if $a=b=c$, that is, if and only if the triangle ABC is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3303. [2008 : 44, 47] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\prod_{\text{cyclic}}\left(2(a+b)^3\right)\ \geq\ \prod_{\text{cyclic}}\left((a+s_1)(bc+s_2)\right),$$

where $s_1=a+b+c$ and $s_2=ab+bc+ca$.

Composite of similar solutions by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain ; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

By straightforward computations, we have

$$\begin{aligned} & 2(a+b)(b+c)(c+a) \ - \ (a+s_1)(bc+s_2) \\ = \ & 2(a+b)(b+c)(c+a) \ - \ (2a+b+c)(ab+2bc+ca) \\ = \ & 2(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc) \\ & \ - \ (2a^2b+ab^2+2b^2c+2bc^2+c^2a+2ca^2+6abc) \\ = \ & ab^2+ac^2-2abc \ = \ a(b-c)^2 \ \geq \ 0\,. \end{aligned}$$

Hence,

$$2(a+b)(b+c)(c+a) \ \geq \ (a+s_1)(bc+s_2)\,.$$

Similarly, we have

$$\begin{aligned} 2(a+b)(b+c)(c+a) \ &\geq \ (b+s_1)(ca+s_2)\,; \\ 2(a+b)(b+c)(c+a) \ &\geq \ (c+s_1)(ab+s_2)\,. \end{aligned}$$

The result now follows by multiplying across the last three inequalities.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece ; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina ; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA ; MICHEL BATAILLE, Rouen, France ; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy ; NGUYEN MANH DUNG, High school student, HUS, Hanoi, Vietnam ; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy ; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan ; PANOS E. TSAOUSSOGLOU, Athens, Greece ; GEORGE TSAPAKIDIS, Agrinio, Greece ; PETER Y. WOO, Biola University, La Mirada, CA, USA ; TITU ZVONARU, Comănești, Romania ; and the proposer.

3304. [2008 : 45, 47] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers and identify a_{n+1} with a_1. Prove that

$$\sum_{k=1}^{n} a_k^3 \ \geq \ \sum_{k=1}^{n} a_k a_{k+1}^2\,.$$

Similar solutions by Michel Bataille, Rouen, France ; Walther Janous, Ursulinengymnasium, Innsbruck, Austria ; Steven Karp, student, University of Waterloo, Waterloo, ON ; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain ; and Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

Reorder the numbers $a_1, a_2, \ldots, a_n$ from smallest to largest and rename them $x_1, x_2, \ldots, x_n$. Then $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and if $y_i = x_i^2$

for each i then we also have $0 \le y_1 \le y_2 \le \cdots \le y_n$. The Rearrangement Inequality states that $\sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} x_i y_{\sigma(i)}$ for any permutation σ of $\{1, 2, \dots, n\}$. Since $\sum_{k=1}^{n} a_k^3 = \sum_{i=1}^{n} x_i y_i$ and $\sum_{k=1}^{n} a_k a_{k+1}^2 = \sum_{i=1}^{n} x_i y_{\sigma(i)}$ for an appropriate permutation σ, the result follows.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, University of Toledo, Toledo, OH, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Ricardo comments that this problem appears as Problem 11.7 on p. 148 of Elementary Inequalities *by D.S. Mitrinović (P. Nordhoff, 1964), but that no solution is provided there.*

3305. [2008 : 45, 47] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Prove that

$$\begin{aligned} \tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} &= \tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13} \\ &= \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13} = \sqrt{13 + 2\sqrt{13}}\,. \end{aligned}$$

Solution by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain, in memory of Jim Totten.

We will prove that

$$\begin{aligned} \tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} &= \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13} \\ &= \tan\frac{6\pi}{13} - 4\sin\frac{5\pi}{13} = \sqrt{13 + 2\sqrt{13}} \end{aligned} \qquad (1)$$

and

$$\begin{aligned} \tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13} &= -\tan\frac{\pi}{13} + 4\sin\frac{3\pi}{13} \\ &= -\tan\frac{3\pi}{13} + 4\sin\frac{4\pi}{13} = \sqrt{13 - 2\sqrt{13}}\,. \end{aligned} \qquad (2)$$

We will make use of two elegant results due to K.F. Gauss and included in the *Sectio VII* of the *Disquisitiones Arithmeticæ* (DA).

Lemma (DA, art. 362, II). Let $n > 1$ be an odd number and $\omega = \dfrac{2k\pi}{n}$, where k is any of the numbers $1, 2, \ldots, n-1$. Then,

$$\tan\omega \;=\; 2\big[\sin(2\omega) - \sin(4\omega) + \sin(6\omega) + \cdots \mp \sin\big((n-1)\omega\big)\big]\,.$$

Theorem (DA, art. 356). Let $n > 1$ be an odd prime number, $\mathfrak{R}$ be the set of the (positive and less than n) quadratic residues modulo n, and $\mathfrak{N}$ be the set of the (positive and less than n) quadratic non-residues modulo n. Then,

$$\sum_{r\in\mathfrak{R}} \cos\frac{2\pi r}{n} - \sum_{m\in\mathfrak{N}} \cos\frac{2\pi m}{n} \;=\; \begin{cases} \sqrt{n} & if \quad n \equiv 1 \pmod 4\ , \\ 0 & if \quad n \equiv 3 \pmod 4\ , \end{cases}$$

and

$$\sum_{r\in\mathfrak{R}} \sin\frac{2\pi r}{n} - \sum_{m\in\mathfrak{N}} \sin\frac{2\pi m}{n} \;=\; \begin{cases} 0 & if \quad n \equiv 1 \pmod 4\ , \\ \sqrt{n} & if \quad n \equiv 3 \pmod 4\ . \end{cases}$$

For $n = 13$ with $\omega = \dfrac{2k\pi}{n}$ and $1 \le k \le 12$ the Lemma yields

$$\tan\omega = 2\big[\sin(2\omega) - \sin(4\omega) + \sin(6\omega) - \sin(8\omega) + \sin(10\omega) - \sin(12\omega)\big]\,.$$

We compute with different values of k in this identity as follows.
If $k = 1$ and $\omega = \dfrac{2\pi}{13}$, then the Lemma yields

$$\tan\frac{2\pi}{13} = 2\left(\sin\frac{4\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{\pi}{13} + \sin\frac{3\pi}{13} - \sin\frac{6\pi}{13} + \sin\frac{2\pi}{13}\right)\,. \tag{3}$$

If $k = 3$ and $\omega = \dfrac{6\pi}{13}$, then the Lemma yields

$$\tan\frac{6\pi}{13} = 2\left(\sin\frac{\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right)\,. \tag{4}$$

If $k = 4$ and $\omega = \dfrac{8\pi}{13}$, then $\tan\dfrac{5\pi}{13} = -\tan\dfrac{8\pi}{13}$ and the Lemma yields

$$\tan\frac{5\pi}{13} = 2\left(\sin\frac{3\pi}{13} + \sin\frac{6\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{\pi}{13} - \sin\frac{2\pi}{13} - \sin\frac{5\pi}{13}\right)\,. \tag{5}$$

By comparing the equations (3), (4), and (5) we see that the first three expressions in equation (1) are equal.
If $k = 2$ and $\omega = \dfrac{4\pi}{13}$, then the Lemma yields

$$\tan\frac{4\pi}{13} = 2\left(\sin\frac{5\pi}{13} + \sin\frac{3\pi}{13} - \sin\frac{2\pi}{13} - \sin\frac{6\pi}{13} - \sin\frac{\pi}{13} + \sin\frac{4\pi}{13}\right)\,. \tag{6}$$

If $k = 5$ and $\omega = \dfrac{10\pi}{13}$, then $\tan\dfrac{3\pi}{13} = -\tan\dfrac{10\pi}{13}$ and the Lemma yields

$$\tan\frac{3\pi}{13} = 2\left(\sin\frac{6\pi}{13} - \sin\frac{\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{4\pi}{13} - \sin\frac{3\pi}{13}\right)\,. \tag{7}$$

If $k = 6$ and $\omega = \dfrac{12\pi}{13}$, then $\tan\dfrac{\pi}{13} = -\tan\dfrac{12\pi}{13}$ and the Lemma yields

$$\tan\frac{\pi}{13} = 2\left(\sin\frac{2\pi}{13} - \sin\frac{4\pi}{13} + \sin\frac{6\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{3\pi}{13} - \sin\frac{\pi}{13}\right) . \quad (8)$$

By comparing the equations (6), (7), and (8) we see that the first three expressions in equation (2) are equal.

Now we take $A = \tan\dfrac{2\pi}{13} + 4\sin\dfrac{6\pi}{13}$ and $B = \tan\dfrac{4\pi}{13} + 4\sin\dfrac{\pi}{13}$. Clearly A and B are positive numbers. From (3) and (6) it follows that

$$A + B = 4\left(\sin\frac{\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13}\right)$$

and

$$A - B = 4\left(\sin\frac{2\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right) .$$

Then,

$$A^2 - B^2 = 16\left(\sin\frac{\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13}\right)\left(\sin\frac{2\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right) .$$

Applying the identity $2\sin a \sin b = \cos(a - b) - \cos(a + b)$, we have

$$A^2 - B^2 = 8\left(\cos\frac{\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{6\pi}{13}\right) .$$

However, for $n = 13 \equiv 1 \pmod 4$, the sets $\mathfrak{R}$ and $\mathfrak{N}$ in the Theorem are $\mathfrak{R} = \{1,4,9,3,12,10\}$ and $\mathfrak{N} = \{2,8,6,11,5,7\}$; thus, by the Theorem

$$2\left(\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13}\right) = \sqrt{13} ,$$

and therefore

$$A^2 - B^2 = 4\sqrt{13} .$$

Similarly, using the identity $2\sin^2 a = 1 - \cos(2a)$ we deduce that

$$\begin{aligned} AB &= 4\left(\sin\frac{\pi}{13} + \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} - \sin\frac{5\pi}{13} + \sin\frac{6\pi}{13}\right) \\ &\qquad \times \left(\sin\frac{\pi}{13} - \sin\frac{2\pi}{13} + \sin\frac{3\pi}{13} + \sin\frac{4\pi}{13} + \sin\frac{5\pi}{13} - \sin\frac{6\pi}{13}\right) \\ &= 6\left(\cos\frac{\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{6\pi}{13}\right) \\ &= \frac{6\sqrt{13}}{2} = 3\sqrt{13} . \end{aligned}$$

For positive real numbers A and B with $A > B$, the solutions of the equations $A^2 - B^2 = 4\sqrt{13}$ and $AB = 3\sqrt{13}$ are

$$A = \sqrt{13 + 2\sqrt{13}} \qquad \text{and} \qquad B = \sqrt{13 - 2\sqrt{13}} .$$

This completes the proof of the identities (1) and (2). The following similar identities can be deduced when $n = 11$:

$$\begin{aligned}\tan\frac{\pi}{11} + 4\sin\frac{3\pi}{11} &= -\tan\frac{2\pi}{11} + 4\sin\frac{5\pi}{11} \\ &= \tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \tan\frac{4\pi}{11} + 4\sin\frac{\pi}{11} \\ &= \tan\frac{5\pi}{11} - 4\sin\frac{4\pi}{11} \\ &= \sqrt{11}\,.\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece (2 solutions); JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

All solvers noted that $\tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13} \neq \sqrt{13 + 2\sqrt{13}}$*, as did George Apostolopoulos, Messolonghi, Greece; and Luyan Zhong-Qiao, Columbia International College, Hamilton, ON. Wagon used Mathematica to check that the first and third identities are correct and the second is incorrect. The proposer offered a partially correct solution.*

Woo wondered if similar results hold for $\frac{\pi}{5}$, $\frac{\pi}{7}$, $\frac{\pi}{11}$ *or* $\frac{\pi}{17}$*. For the case of* $\frac{\pi}{11}$ *Benito et al. answered (above) in the affirmative. The interested reader may want to investigate the other cases. Woo also challenges the readers to find geometric proofs for the equalities in (1) and (2).*

3306. [2008 : 45, 47] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Find a real number t and polynomials $f(x)$, $g(x)$, and $h(x)$ with integer coefficients, such that

$$f(t) = \sqrt{2}\,, \qquad g(t) = \sqrt{3}\,, \qquad \text{and} \qquad h(t) = \sqrt{7}\,.$$

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

Set $\theta = \sqrt{2} + \sqrt{3} + \sqrt{7}$ and $\psi = \sqrt{2}\sqrt{3}\sqrt{7}$. Computing θ^3, θ^5, and θ^7, we obtain

$$\begin{aligned}\sqrt{2} + \sqrt{3} + \sqrt{7} &= \theta\,, \\ 16\sqrt{2} + 15\sqrt{3} + 11\sqrt{7} + 3\psi &= \frac{1}{2}\theta^3\,, \\ 281\sqrt{2} + 241\sqrt{3} + 161\sqrt{7} + 60\psi &= \frac{1}{4}\theta^5\,, \\ 4796\sqrt{2} + 3975\sqrt{3} + 2611\sqrt{7} + 1043\psi &= \frac{1}{8}\theta^7\,.\end{aligned}$$

This is a linear system for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, and ψ. Solving for $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{7}$

yields

$$\sqrt{2} = \frac{59}{20}\theta - \frac{1}{2}\theta^3 + \frac{1}{80}\theta^5 ,$$
$$\sqrt{3} = \frac{313}{80}\theta - \frac{297}{160}\theta^3 + \frac{67}{320}\theta^5 - \frac{3}{640}\theta^7 ,$$
$$\sqrt{7} = -\frac{469}{80}\theta + \frac{377}{160}\theta^3 - \frac{71}{320}\theta^5 + \frac{3}{640}\theta^7 .$$

Finally, setting $t = \frac{\theta}{80} = \frac{\sqrt{2}+\sqrt{3}+\sqrt{7}}{80}$ we obtain $\sqrt{2} = f(t)$, $\sqrt{3} = g(t)$, and $\sqrt{7} = h(t)$, where the polynomials $f(x)$, $g(x)$, and $h(x)$ have integer coefficients :

$$f(x) = 236x - \frac{1}{2}(80)^3x^3 + (80)^4x^5 ,$$
$$g(x) = 313x - \frac{1}{2}(80)^2 \cdot 297x^3 + \frac{1}{4}(80)^4 \cdot 67x^5 - \frac{3}{8}(80)^6x^7 ,$$
$$h(x) = -469x + \frac{1}{2}(80)^2 \cdot 377x^3 - \frac{1}{4}(80)^4 \cdot 71x^5 + \frac{3}{8}(80)^6x^7 .$$

Also solved by MOHAMMED AASSILA, Strasbourg, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3307. [2008 : 45, 47] *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Eliminate θ from the system

$$\lambda\cos(2\theta) = \cos(\theta+\alpha) ,$$
$$\lambda\sin(2\theta) = 2\sin(\theta+\alpha) .$$

Similar solutions by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain; Joe Howard, Portales, NM, USA; and George Tsapakidis, Agrinio, Greece.

The given system can be rewritten as a linear system in $\cos\alpha$ and $\sin\alpha$:

$$\cos\theta\cos\alpha - \sin\theta\sin\alpha = \lambda(\cos^2\theta - \sin^2\theta) ,$$
$$\sin\theta\cos\alpha + \cos\theta\sin\alpha = \lambda\sin\theta\cos\theta .$$

Its solution is then

$$\cos\alpha = \lambda\cos^3\theta , \quad \text{and} \quad \sin\alpha = \lambda\sin^3\theta ;$$

whence,

$$(\cos\alpha)^{2/3} + (\sin\alpha)^{2/3} \;=\; \lambda^{2/3}\,. \tag{1}$$

Comments from the Spanish team. Note that the original system has a solution if and only if α and λ satisfy (1). In particular, letting $x = \cos\alpha$ and $y = \sin\alpha$, we see that for $1 \le |\lambda| \le 2$ the solutions can be represented by the intersection points of the unit circle $x^2 + y^2 = 1$ with the astroid $x^{2/3} + y^{2/3} = \lambda^{2/3}$. Thus for each λ with absolute value between 1 and 2, (1) will be satisfied for eight values of α; for $\lambda \in \{\pm 1, \pm 2\}$, it will be satisfied by four values of α. There can be no real solutions for other values of λ.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; ANDREA MUNARO, student, University of Trento, Trento, Italy; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; BOB SERKEY, Leonia, NJ, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was one incorrect submission.

Our readers produced solutions in a variety of formats. Here are a few of the nicest. Instead of (1), Alt, Bataille, and the proposer independently obtained the equivalent equation

$$\sin^2(2\alpha) \;=\; \frac{4\left(\lambda^2 - 1\right)^3}{27\lambda^2}\,.$$

Geupel found that in terms of a real parameter t, the solutions of the given system satisfy

$$\theta \;=\; \arctan t + m\pi\,, \quad \alpha \;=\; \arctan(t^3) + n\pi\,, \quad \textit{and} \quad \lambda \;=\; (-1)^{m+n}\sqrt{\frac{(1+t^2)^3}{1+t^6}}\,,$$

for integers m and n. In addition, there were several implicit solutions where the solver simply presented an equation for θ in terms of α or λ; for example, $\theta = \arctan\sqrt[3]{\tan\alpha} + k\pi$ came from Arslanagić and from Ros.

3308. [2008 : 45, 48] *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Given $\triangle ABC$, let AD be the altitude to BC. If $AB : AC = 1 : \sqrt{3}$, prove that $AD \le \frac{\sqrt{3}}{2}BC$. When does equality hold?

I. *Solution by Joe Howard, Portales, NM, USA.*

It suffices to take $AB = 1$ and $AC = \sqrt{3}$; consequently, $AD = \sin B$. Writing $a = BC$, we therefore must show that

$$\sin B \;\le\; \frac{\sqrt{3}}{2}a\,.$$

We start with the inequality

$$\begin{aligned} 1 &\geq \sin(60^\circ + A) = \sin 60^\circ \cos A + \sin A \cos 60^\circ \\ &= \frac{\sqrt{3}}{2}\cos A + \frac{1}{2}\sin A\,, \end{aligned}$$

which is equivalent to

$$2 - \sqrt{3}\cos A \geq \sin A\,. \tag{1}$$

By the cosine law, $a^2 = 4 - 2\sqrt{3}\cos A$, so the inequality (1) is equivalent to

$$a^2 \geq 2\sin A\,. \tag{2}$$

By the sine law, $\dfrac{\sin A}{a} = \dfrac{\sin B}{\sqrt{3}}$, or

$$a = \frac{\sqrt{3}\sin A}{\sin B}\,,$$

whence (2) is equivalent to

$$a\frac{\sqrt{3}\sin A}{\sin B} \geq 2\sin A\,,$$

which reduces immediately to what we were to show. Equality occurs when $60^\circ + A = 90^\circ$, in which case $A = 30^\circ$ and $a^2 = 4 - 2\sqrt{3}\cos 30^\circ = 1$. Thus, equality holds if and only if $\triangle ABC$ is isosceles with $A = C = 30^\circ$.

II. *Solution by Michel Bataille, Rouen, France.*

We are given that vertex A belongs to the locus of those points P for which $\dfrac{PB}{PC} = \dfrac{1}{\sqrt{3}}$. We recognize this locus to be a circle Γ called the *circle of Apollonius*; it intersects symmetrically the line joining B to C in points that divide the segment BC internally and externally in the ratio $1 : \sqrt{3}$. As an easy consequence, the centre K of Γ satisfies $\overrightarrow{BK} = -\frac{1}{2}\overrightarrow{BC}$, and its radius is $\dfrac{\sqrt{3}BC}{2}$. Let NN' be the diameter of Γ that is perpendicular to BC. Then N and N' are the points of Γ that are farthest from the line BC, hence

$$AD = d(A, BC) \leq d(N, BC) = \frac{\sqrt{3}BC}{2}\,.$$

Equality holds if and only if A is situated at N or N', in which case $\triangle ABC$ is isosceles with $BA = BC$ and $\angle ABC = 120^\circ$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, *MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; STEVEN KARP, student, University of Waterloo, Waterloo, ON; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

More generally, Konečný proved that $AD \leq \frac{q}{q^2-1}BC$ *if* $AC : AB = q$ *with* $q > 1$. *His argument was much like that of solution II above.*

3309. [2008 : 45, 48] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let α, β, and γ be fixed non-zero real numbers. Show that the system

$$\begin{aligned} \alpha x + \beta y + \gamma z &= 1\,, \\ xy + yz + zx &= 1\,, \end{aligned}$$

has a unique solution for (x, y, z) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 + 1 \;=\; 2(\alpha\beta + \beta\gamma + \gamma\alpha)\,,$$

and, in this case find that unique solution.

Solution by George Tsapakidis, Agrinio, Greece, modified by the editor.

Substituting $\alpha x = 1 - \beta y - \gamma z$ into $\alpha(xy + yz + zx) = \alpha$ we obtain $(y+z)(1-\beta y-\gamma z)+\alpha yz = \alpha$, which upon simplifying yields the following quadratic equation in y

$$\beta y^2 - \left[1 + (\alpha - \beta - \gamma)z\right]y + \gamma z^2 - z + \alpha \;=\; 0\,. \tag{1}$$

Equation (1) has a unique solution in y if and only if

$$\left[1 + (\alpha - \beta - \gamma)z\right]^2 - 4\beta(\gamma z^2 - z + \alpha) \;=\; 0\,,$$

which can be written as the following quadratic equation in z

$$\left[(\alpha - \beta - \gamma)^2 - 4\beta\gamma\right] z^2 \;+\; 2(\alpha + \beta - \gamma)z + 1 - 4\alpha\beta \;=\; 0\,. \tag{2}$$

Equation (2) has a unique solution in z if and only if

$$(\alpha + \beta - \gamma)^2 - \left[(\alpha - \beta - \gamma)^2 - 4\beta\gamma\right](1 - 4\alpha\beta) \;=\; 0\,,$$

which, by straightforward computations, reduces to

$$\alpha^2 + \beta^2 + \gamma^2 + 1 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)\,. \tag{3}$$

Now, suppose (3) holds. Then we have

$$(\alpha-\beta-\gamma)^2-4\beta\gamma \;=\; \alpha^2+\beta^2+\gamma^2-2\alpha\beta-2\beta\gamma-2\gamma\alpha \;=\; -1\,.$$

Thus, (2) reduces to $-z^2+2(\alpha+\beta-\gamma)z+1-4\alpha\beta=0$, the unique solution of which is given by $z=\alpha+\beta-\gamma$.

Using this and (3), we find that

$$\begin{aligned} y &= \frac{1+(\alpha-\beta-\gamma)(\alpha+\beta-\gamma)}{2\beta} = \frac{1+(\alpha-\gamma)^2-\beta^2}{2\beta} \\ &= \frac{1+\alpha^2-\beta^2+\gamma^2-2\alpha\gamma}{2\beta} = \frac{-2\beta^2+2\alpha\beta+2\beta\gamma}{2\beta} = \gamma+\alpha-\beta\,. \end{aligned}$$

Finally, using (3) we have

$$\begin{aligned} \alpha x &= 1-\beta y-\gamma z \;=\; 1-\beta(\gamma+\alpha-\beta)-\gamma(\alpha+\beta-\gamma) \\ &= -\alpha^2+\alpha\beta+\gamma\alpha\,, \end{aligned}$$

from which it follows that $x=\beta+\gamma-\alpha$.

Therefore, the unique solution to the given system is

$$(x,y,z) \;=\; (\beta+\gamma-\alpha,\; \gamma+\alpha-\beta,\; \alpha+\beta-\gamma)\,.$$

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; STEVEN KARP, student, University of Waterloo, Waterloo, ON; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was also one partially incorrect solution submitted.

3310. [2008 : 46, 48] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b, and c denote, as usual, the lengths of the sides BC, CA, and AB, respectively, in $\triangle ABC$. Let s be the semiperimeter of $\triangle ABC$, r the inradius, h_a the altitude to side BC, and r_a, r_b, and r_c the exradii to A, B, and C, respectively.

(a) Show that for $x>0$, we have $h_a=\dfrac{2s(s-a)x}{x^2+s(s-a)}$ if and only if $x=r_b$ or $x=r_c$.

(b) Show that for $x>0$, we have $h_a=\dfrac{2(s-b)(s-c)x}{|x^2-(s-b)(s-c)|}$ if and only if $x=r$ or $x=r_a$.

Similar solutions by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Titu Zvonaru, Comăneşti, Romania; and the proposer.

It is well known that the area F of $\triangle ABC$ can be variously expressed as $\frac{1}{2}ah_a$, rs, $r_a(s-a)$, $r_b(s-b)$, $r_c(s-c)$, or $\sqrt{s(s-a)(s-b)(s-c)}$. We have

$$\begin{aligned} r_b + r_c &= \frac{F}{s-b} + \frac{F}{s-c} = \frac{F(2s-b-c)}{(s-b)(s-c)} \\ &= \frac{a\cdot F\cdot(s-a)}{s(s-a)(s-b)(s-c)} = \frac{as(s-a)}{F} \\ &= \frac{2as(s-a)}{ah_a} = \frac{2s(s-a)}{h_a} \end{aligned} \qquad (1)$$

and

$$r_b r_c = \frac{F^2}{(s-b)(s-c)} = \frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)} = s(s-a)\,. \qquad (2)$$

It follows from (1) and (2) that $x = r_b$ and $x = r_c$ are the solutions of

$$h_a x^2 - 2s(s-a)x + h_a s(s-a) = 0\,,$$

hence these are the solutions of the equation in (a).

We also have

$$\begin{aligned} r_a - r &= \frac{F}{s-a} - \frac{F}{s} = \frac{aF}{s(s-a)} \\ &= \frac{a\cdot F\cdot(s-b)(s-c)}{F^2} = \frac{2(s-b)(s-c)}{h_a} \end{aligned} \qquad (3)$$

and

$$r_a r = \frac{F^2}{s(s-a)} = (s-b)(s-c)\,. \qquad (4)$$

By (3) and (4) it follows that the equation

$$h_a x^2 - 2(s-b)(s-c)x - (s-b)(s-a)h_a = 0$$

has the solutions $x = r_a$ and $x = -r$ and that the equation

$$h_a x^2 + 2(s-b)(s-c)x - (s-b)(s-a)h_a = 0$$

has the solutions $x = -r_a$ and $x = r$. Thus, the only positive solutions of the preceding two equations are r and r_a, and it follows that these are the only positive solutions to the equation in (b).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3311. [2008 : 46, 48] *Proposed by Michel Bataille, Rouen, France.*

Let n be an integer with $n \geq 2$. Suppose that for $k = 0, 1, \ldots, n-2$ we have

$$\binom{n-2}{k} \equiv (-1)^k(k+1) \pmod{n} .$$

Show that n is a prime.

Solution by Oliver Geupel, Brühl, NRW, Germany.

It is sufficient to prove that if n is a composite integer with $n \geq 2$, then there exists an integer k with $0 \leq k \leq n-2$ such that

$$\binom{n-2}{k} \not\equiv (-1)^k(k+1) \pmod{n} . \qquad (1)$$

Toward that end, let p be the least prime factor of n. If (1) holds for some $k < p$, then we are done. Otherwise the given congruence holds in particular for $k = p-2$, and we have

$$\begin{aligned} (p-1)\binom{n-2}{p} &= \binom{n-2}{p-2} \cdot \frac{n-p}{p} \cdot (n-p-1) \\ &\equiv (-1)^{p-2}(p-1) \cdot \left(\frac{n}{p} - 1\right) \cdot (-p-1) \\ &\equiv (p-1) \cdot (-1)^p \left(p + 1 - \frac{n}{p}\right) \pmod{n} . \end{aligned}$$

Because $p-1$ is coprime to the modulus n, we can divide both sides by it and conclude that (1) holds with $k = p$:

$$\binom{n-2}{p} \equiv (-1)^p \left(p + 1 - \frac{n}{p}\right) \not\equiv (-1)^p(p+1) \pmod{n} .$$

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; STEVEN KARP, student, University of Waterloo, Waterloo, ON; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

The converse is a known result : If p is a prime, then $\binom{p-2}{k} \equiv (-1)^k(k+1) \pmod{p}$ for $k = 0, 1, \ldots, p-2$. *Karp provided a simple proof (by induction on k); Bataille provided a reference : E. Lucas,* Théorie des nombres, *A. Blanchard (1961), p. 420.*

3312. [2008 : 46, 48] *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer congruent to 1 modulo 6. Show that $3/n$ can be expressed as

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$$

for some distinct positive integers $a_1, a_2, \ldots, a_k$, and find the minimal value of k.

Solution by Steven Karp, student, University of Waterloo, Waterloo, ON.

We solve the problem for all positive integers n. If $n \equiv 0$ or $3 \pmod 6$, then $\frac{3}{n} = \frac{1}{n/3}$. If $n \equiv -1 \pmod 6$, then

$$\frac{3}{n} = \frac{1}{\frac{n+1}{3}} + \frac{1}{\frac{n(n+1)}{3}}\,.$$

If $n \equiv \pm 2 \pmod 6$, then

$$\frac{3}{n} = \frac{1}{\frac{n}{2}} + \frac{1}{n}\,.$$

If $n \equiv 1 \pmod 6$ and n has a factor c such that $c \equiv -1 \pmod 6$, then

$$\frac{3}{n} = \frac{1}{\frac{n+c}{3}} + \frac{1}{\frac{n(n+c)}{3c}}\,.$$

We see that k is minimal in all of these cases, since $\frac{3}{n} = \frac{1}{a_1}$ for some positive integer a_1 only if $n \equiv 0 \pmod 3$. Now, if $n > 1$, $n \equiv 1 \pmod 6$ and all divisors of n are congruent to 1 modulo 6, then

$$\frac{3}{n} = \frac{1}{\frac{n+1}{2}} + \frac{1}{n} + \frac{1}{\frac{n(n+1)}{2}}\,,$$

and we claim that $k = 3$ is minimal in this case. To prove this suppose for the sake of contradiction that

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b}$$

for distinct positive integers a and b. Then

$$n = \frac{b(3a-n)}{a}\,,$$

and we have $a|b(3a-n)$. Therefore, $a = pq$ where p and q are positive integers such that $p|b$ and $q|(3a-n)$. Then $\frac{b}{p}|n$, whence $\frac{b}{p} \equiv 1 \pmod 6$. Since $q|a$ and $q|(3a-n)$, we also have $q|n$, so $q \equiv 1 \pmod 6$. We now have

$$1 \equiv qn = \left(\frac{b}{p}\right)(3a-n) \equiv 3a - n \equiv -1 \text{ or } 2 \pmod 6\ ,$$

a contradiction.

Finally, for $n = 1$ we have

$$3 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{15} + \frac{1}{230} + \frac{1}{57960}\,.$$

A computer search shows that $k = 13$ is minimal in this case.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Benito et al. refer to the note by Thomas R. Hagedorn, A proof of a conjecture on Egyptian fractions, Amer. Math. Monthly, **107** *(2000) 62-63, where it is proved that for each odd integer* $n \geq 3$ *not divisible by* 3 *there exist distinct odd, positive integers* a, b, *and* c *such that*

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\,.$$

The result had been conjectured by R. Hardin and N. Sloane. When $n = 6p+1$ *Hagedorn gives the decomposition*

$$\frac{3}{n} = \frac{3}{6p+1} = \frac{1}{2p+1} + \frac{1}{(2p+1)(4p+1)} + \frac{1}{(4p+1)(6p+1)}\,.$$

Janous refers to a paper by Andrzej Schinzel, Sur quelques propriétés des nombres $3/n$ *et* $4/n$*, où n est un nombre impair,* Mathesis *65 (1956) 219-222, for a treatment of our problem. Since the improper fraction* $3 = 3/1$ *arises, he asks if any **CRUX** readers know the minimum number* $\ell(n)$ *of distinct Egyptian fractions needed to represent the positive integer* n.

A Happy New Year to all ***CRUX with MAYHEM*** readers. The Jim Totten special issue was slated to be completed in May of this year, but due to delays we are now going to release the special issue in September 2009 instead.

This year we plan on improving our database of names and affiliations of you, the readers. If your name does not look quite right, for example, if the accents are not quite right or your family name is incorrect, etc., then please let us know and we will update our files. Many international readers subscribe to ***CRUX with MAYHEM*** and we want to get these (fascinating !) details right.

Václav Linek.