Problem of the Month

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Here is a problem that requires only some careful reasoning (albeit pretty tricky careful reasoning) and the ability to add.

Problem (2006 Grade 8 Gauss Contest)

In the diagram, the numbers from 1 to 25 are to be arranged in the 5×5 grid so that each number, except 1 and 2, is the sum of two of its neighbours. (Numbers in the grid are *neighbours* if their squares touch along a side or at a corner. For example, the "1" has 8 neighbours.) Some of the numbers have already been filled in. Which number must replace the "?" when the grid is completed?

			20	**21**
	6	**5**	**4**	
23	**7**	**1**	**3**	?
	9	**8**	**2**	
25	**24**			**22**

This is not another Sudoku—honest! It looks a bit like one, though. That is part of the reason why this problem was included on the Contest—it is nice to have problems that look familiar but, upon closer examination, are a bit different.

Solution: We could just fiddle around by trial and error until we get some numbers that work. But we will walk through the solution in a logical way.

It's tough to know exactly where to start. First, it makes sense to check which numbers are missing. The grid already includes the numbers 1 to 9 and 20 to 25; so those missing are 10 to 19.

Next, we could figure out which numbers in the grid are already the sum of two neighbours. For example, 9 has neighbours 1 and 8 (and $9 = 1 + 8$); 8 has neighbours 1 and 7 (and $8 = 1 + 7$), and so on. Let's italicize every number which is already the sum of two of its neighbours, as well as the entries 1 and 2.

			20	**21**
	6	*5*	*4*	
23	*7*	*1*	*3*	?
	9	*8*	*2*	
25	**24**			**22**

Now what? It's probably time for that tried and true problem-solving technique—panic. After we get that out of our system, we might try looking at some of the numbers that have almost all of their neighbours already filled in. Also, we might as well focus on the part of the grid near the "?".

For example, consider 21. Since 21 already has neighbours 20 and 4, we must write 21 as either $20 + 1$ or $4 + 17$. But the number 1 already appears elsewhere in the grid; thus, the empty space below 21 must be 17.

			20	*21*
	6	*5*	*4*	**17**
23	*7*	*1*	*3*	?
	9	*8*	*2*	
25	**24**			**22**

Looking at 17 as we did with 21, we see that 17 must be $3 + 14$ or $4 + 13$; thus, the "?" must represent either 13 or 14. But we can't say for sure yet which one it is.

How about 22? It cannot be $2 + 20$, as 20 is already accounted for. What two numbers add to 22 and are not yet in the grid? The only possibility is 10 and 12, in some order. But can we tell which of 10 and 12 is placed where? If 10 was above 22, we could not get 10 as the sum of two neighbours, since $2 + 8$ and $3 + 7$ are not possible. If 12 is above 22, then $12 = 10 + 2$ and $10 = 8 + 2$, which can work.

			20	*21*
	6	*5*	*4*	**17**
23	*7*	*1*	*3*	?
	9	*8*	*2*	*12*
25	**24**		*10*	*22*

We know that the "?" is either 13 or 14. Could it be 13? Are there two neighbours of "?" that add to 13? No. So the "?" must be 14, which solves the problem.

But wait! We can't stop now! Let's carry on a bit further.

Looking at 25, we see that 25 must be $24 + 1$ (not a possibility) or $9 + 16$. Hence, the number in the space above 25 must be 16. This now allows us to italicize 23, 24, 25, and 16. (Why?)

			20	*21*
	6	*5*	*4*	*17*
23	*7*	*1*	*3*	*14*
16	*9*	*8*	*2*	*12*
25	*24*		*10*	*22*

Try completing the rest of the grid on your own!