

Mayhem Solutions

M257. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

For a given positive integer k , consider the set of lattice points $\{(x, y)\}$ where x and y are integers such that $0 \leq x \leq 2k + 1$ and $0 \leq y \leq 2k + 1$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0, 0)$ is an integer (possibly 0).

Solution by Hasan Denker, Istanbul, Turkey.

This problem is a generalization of Mayhem problem M253 in which case k was equal to 3, and can be solved in a similar fashion. Noting that the probability that a randomly selected integer between 0 and $2k + 1$ is even (or odd) is $\frac{1}{2}$, and using a similar argument as for M253, we find that the probability that the area of the triangle is an integer is $\frac{5}{8}$. We can therefore conclude that the probability that the area of the triangle is an integer is independent of k .

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Beaverton, OR, USA; and the proposer. One incorrect solution was also submitted.

M258. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let c , d , and n be integers such that $n = c^2 + d^2$. Prove that $n = (a^2 + b^2)/5$ for some integers a and b .

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

Take $a = 2c - d$ and $b = 2d + c$. Since c and d are integers, it follows that a and b are also integers. We then have

$$\frac{a^2 + b^2}{5} = \frac{(2c - d)^2 + (2d + c)^2}{5} = \frac{5(c^2 + d^2)}{5} = c^2 + d^2 = n.$$

Hence, such integers exist by construction.

Also solved by ARKADY ALT, San Jose, CA, USA; HOUDA ANOUN, Bordeaux, France; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; and D. KIPP JOHNSON, Beaverton, OR, USA.

M259. *Proposed by the Mayhem Staff.*

The number n is formed by concatenating the strings of digits formed by the numbers 2^{2006} and 5^{2006} . How many digits does n have?

Solution by Arkady Alt, San Jose, CA, USA.

More generally, for any natural number m , let p and q be the number of digits in the strings of digits formed by 2^m and 5^m , respectively. Then $10^{p-1} < 2^m < 10^p$ and $10^{q-1} < 5^m < 10^q$. Therefore,

$$(10^{p-1})(10^{q-1}) < 2^m \cdot 5^m < 10^p \cdot 10^q;$$

that is,

$$10^{p+q-2} < 10^m < 10^{p+q}.$$

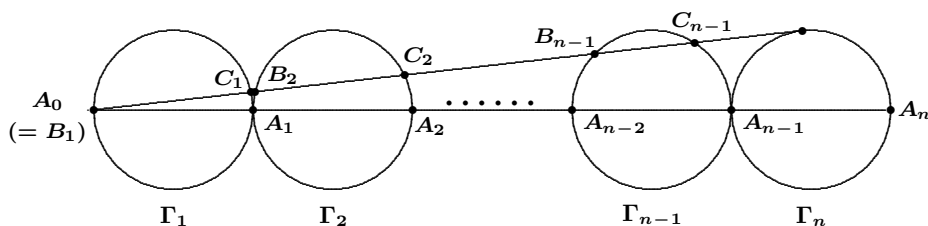
Thus, $p + q - 2 < m < p + q$, which is equivalent to $m = p + q - 1$, or $p + q = m + 1$. We can conclude that a concatenation of 2^m and 5^m has $m + 1$ digits. In particular, taking $m = 2006$, we find that n has 2007 digits.

Also solved by HOUDA ANOUN, Bordeaux, France; ALPER CAY, Uzman Private School, Kayseri, Turkey; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; D. KIPP JOHNSON, Beaverton, OR, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M260. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

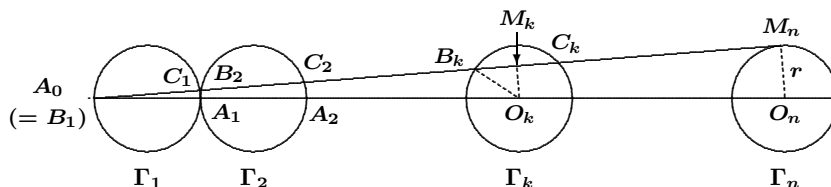
Points A_0, A_1, \dots, A_n lie on a line, in that order, spaced a uniform distance $2r$ apart. For $1 \leq k \leq n$, let Γ_k be the circle with $A_{k-1}A_k$ as diameter. The line through A_0 tangent to Γ_n intersects the circle Γ_k at the points B_k and C_k , for $1 \leq k \leq n-1$.

Determine the length of the line segment B_kC_k for $1 \leq k \leq n-1$.



Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let O_k be the centre of Γ_k . Let M_k be the mid-point of chord B_kC_k for $1 \leq k \leq n-1$, and let M_n be the point of tangency to Γ_n .



Let k be fixed such that $1 \leq k \leq n - 1$. It can be seen that triangles $A_0O_kM_k$ and $A_0O_nM_n$ are similar. We can conclude that

$$\frac{O_kM_k}{r} = \frac{A_0O_k}{A_0O_n} = \frac{2kr - r}{2nr - r} = \frac{2k - 1}{2n - 1}.$$

If we set $\theta_k = \angle B_kO_kM_k$, then $\cos \theta_k = \frac{O_kM_k}{r} = \frac{2k - 1}{2n - 1}$. Since $\frac{1}{2}B_kC_k = r \sin \theta_k$, we have

$$\begin{aligned} B_kC_k &= 2r \sin \theta_k = 2r \sqrt{1 - \cos^2 \theta_k} \\ &= 2r \sqrt{1 - \left(\frac{2k - 1}{2n - 1}\right)^2} = \frac{2r}{2n - 1} \sqrt{(2n - 1)^2 - (2k - 1)^2} \\ &= \frac{4r}{2n - 1} \sqrt{(n - k)(n + k - 1)}. \end{aligned}$$

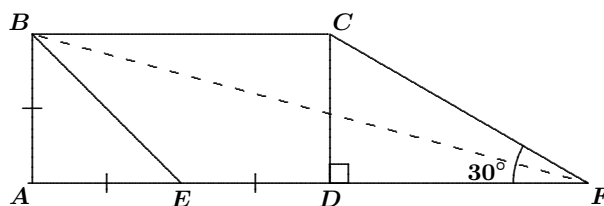
Also solved by KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were two incorrect solutions submitted.

M261. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

Rectangle $ABCD$ has $AB = \frac{1}{2}BC$. On the outside of the rectangle, draw $\triangle DCF$, where $\angle DFC = 30^\circ$ and ADF is a straight line segment. Let E be the mid-point of AD .

Determine the measure of $\angle EBF$.

Essentially the same solution by ROBERT BILINSKI, Collège Montmorency, Laval, QC; ALPER CAY, Uzman Private School, Kayseri, Turkey; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.



In right triangle CDF , we have $\angle DFC = 30^\circ$ and $\angle DCF = 60^\circ$. We can then conclude that $CF = 2CD = 2AB = BC$. Now, considering isosceles triangle BCF , we have $\angle BCF = 150^\circ$ and consequently, $\angle CBF = \angle CFB = 15^\circ$. Also, we know that triangle ABE is isosceles with $\angle ABE = \angle AEB = 45^\circ$. Thus,

$$\angle EBF = 90^\circ - \angle ABE - \angle CBF = 30^\circ.$$

Also solved by COURTIS G. CHRYSOSTOMOS, Larissa, Greece; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; D. KIPP JOHNSON, Beaverton, OR, USA; and GEOFFREY A. KANDALL, Hamden, CT, USA.

M262. Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.

Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ for which $f(1) = 1$ and, for all real numbers x and y , we have $f(x + y) = 3^y f(x) + 2^x f(y)$.

Combination of similar solutions by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Houda Anoun, Bordeaux, France; Hasan Denker, Istanbul, Turkey; Jean-David Houle, student, McGill University, Montreal, QC; D. Kipp Johnson, Beaverton, OR, USA; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let f be any function satisfying the given conditions $f(1) = 1$ and, for all real numbers x and y ,

$$f(x + y) = 3^y f(x) + 2^x f(y). \quad (1)$$

Setting $y = 1$ in (1) gives, for all $x \in \mathbb{R}$,

$$f(x + 1) = 3f(x) + 2^x f(1) = 3f(x) + 2^x. \quad (2)$$

Setting $x = 1$ in (1) gives, for all $y \in \mathbb{R}$,

$$f(1 + y) = 3^y f(1) + 2f(y) = 3^y + 2f(y). \quad (3)$$

Changing y to x in (3), we get, for all $x \in \mathbb{R}$,

$$f(1 + x) = 3^x + 2f(x). \quad (4)$$

Finally, using (2) and (4) and noting that $f(x + 1) = f(1 + x)$, we get

$$3f(x) + 2^x = 3^x + 2f(x).$$

Thus, $f(x) = 3^x - 2^x$.

Also solved by COURTIS G. CHRYSOSTOMOS, Larissa, Greece; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.