MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier février 2008**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M307. *Proposé par Neven Jurič, Zagreb, Croatie.*

Deux carrés magiques 4×4 ont la propriété que la somme de chacune de leurs lignes, de chacune de leurs colonnes et de leurs deux diagonales donne le même nombre N. On considère alors, pour chaque carré, la somme des éléments de ses quatre coins. Ces sommes peuvent-elles être différentes ou doivent-elles être égales ? (En d'autre termes, la somme des éléments des quatre coins dépend-elle du carré lui-même ou de la *somme magique* N ?) Déterminer cette somme si elle est constante, ou alors montrer que ces sommes peuvent différer.

M308. *Proposé par Babis Stergiou, Chalkida, Grèce.*

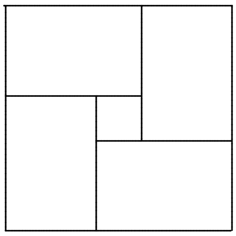
Soit ABC un triangle rectangle avec $A = 90°$, et soit M le point milieu du côté AB. Si D est le pied de la perpendiculaire de A sur CM et N le point milieu de DC, montrer que $BD \perp AN$.

M309. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Déternimer tous les entiers non négatifs possibles x, y, z et t de sorte que $3^x + 3^y + 3^z + 3^t$ soit un cube parfait.

M310. *Proposé par J. Walter Lynch, Athens, GA, É-U.*

Quatre rectangles congruents sont disposés pour former un carré de telle sorte qu'ils entourent un carré plus petit.



Soit S l'aire du carré extérieur et Q celle du carré intérieur. Si l'aire du carré extérieur est 9 fois celle du carré intérieur, déterminer le rapport des côtés des rectangles.

M311. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels positifs, et soit $m \in \left(0, \frac{1}{4}\right)$. Montrer qu'une au moins des équations suivantes possède des solutions réelles :

$$\begin{aligned} ax^2 + bx + cm &= 0\,, \\ bx^2 + cx + am &= 0\,, \\ cx^2 + ax + bm &= 0\,. \end{aligned}$$

M312. *Proposé par G.P. Henderson, Garden Hill, Campbellcroft, ON.*

Jean est en négotiation avec son banquier sur les termes d'une hypothèque. Ils sont tombés d'accord sur le montant L de celle-ci ainsi que sur un taux annuel d'intérêt de i.

Jean propose «Je veux faire des paiements de P dollars à la fin de chaque année pour les prochaines n années. C'est plus qui n'en faut pour payer les intérêts. L'excédent servira à réduire le principal pour l'année suivante. A la fin des n années, je contracterai une nouvelle hypothèque pour le principal restant.»

Le banquier répond «Je préférerais des paiements plus fréquents. Je suggère des paiements de $P/4$ chaque trimestre avec un intérêt de $i/4$ appliqué sur le solde du trimestre précédent.»

Mais Jean s'objecte «Mais alors le taux annuel effectif sera plus grand que i !»

Le banquier rétorque «Oui, mais le montant restant au temps n sera plus petit !»

Jean trouve cela dur à croire. Est-ce vrai ?

. . . .

M307. *Proposed by Neven Jurič, Zagreb, Croatia.*

Two 4×4 magic squares have the property that all four of their rows, all four of their columns, and their two diagonals all sum to the same value N. Consider the sum of the four corner elements of each square. Can these sums be different, or must they be the same? (In other words, does the corner sum depend on the square itself, or only on the *magic sum* N?) Either determine the constant sum, or show that these sums can differ.

M308. *Proposed by Babis Stergiou, Chalkida, Greece.*

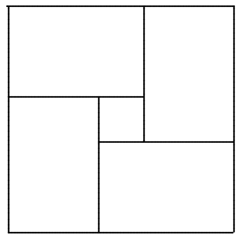
Let ABC be a right triangle with $A = 90^\circ$, and let M be the mid-point of side AB. If D is the foot of the perpendicular from A to CM and N is the mid-point of DC, prove that $BD \perp AN$.

M309. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all possible non-negative integers x, y, z, and t such that $3^x + 3^y + 3^z + 3^t$ is a perfect cube.

M310. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Four congruent rectangles are arranged in a square pattern so that they enclose a smaller square.



Let S be the area of the outer square and Q the area of the inner square. If the area of the outer square is 9 times the area of the inner square, determine the ratio of the sides of the rectangles.

M311. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers, and let $m \in \left(0, \frac{1}{4}\right)$. Show that at least one of the following equations has real roots:

$$\begin{aligned} ax^2 + bx + cm &= 0\,, \\ bx^2 + cx + am &= 0\,, \\ cx^2 + ax + bm &= 0\,. \end{aligned}$$

M312. *Proposed by G.P. Henderson, Garden Hill, Campbellcroft, ON.*

John is negotiating the terms of a mortgage with his bank manager. They have agreed that the loan will be for L dollars and that the annual interest rate will be i.

John says, "I will make payments of P dollars at the end of each year for the next n years. This is more than enough to pay the interest. The excess will reduce the principal outstanding for the next year. At the end of n years, I will arrange a new mortgage for the remaining principal."

The manager responds, "I would like more frequent payments. I suggest payments of $P/4$ each quarter-year with interest rate $i/4$ applied to the previous quarter's balance."

John objects, "But then the effective annual interest rate will be greater than i!"

The manager replies, "Yes, but the amount outstanding at time n will be less!"

John finds this hard to believe. Is it true?

Mayhem Solutions

M257. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

For a given positive integer k, consider the set of lattice points $\{(x, y)\}$ where x and y are integers such that $0 \leq x \leq 2k+1$ and $0 \leq y \leq 2k+1$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0, 0)$ is an integer (possibly 0).

Solution by Hasan Denker, Istanbul, Turkey.

This problem is a generalization of Mayhem problem M253 in which case k was equal to 3, and can be solved in a similar fashion. Noting that the probability that a randomly selected integer between 0 and $2k+1$ is even (or odd) is $\frac{1}{2}$, and using a similar argument as for M253, we find that the probability that the area of the triangle is an integer is $\frac{5}{8}$. We can therefore conclude that the probability that the area of the triangle is an integer is independent of k.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Beaverton, OR, USA; and the proposer. One incorrect solution was also submitted.

M258. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let c, d, and n be integers such that $n = c^2 + d^2$. Prove that $n = (a^2 + b^2)/5$ for some integers a and b.

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

Take $a = 2c - d$ and $b = 2d + c$. Since c and d are integers, it follows that a and b are also integers. We then have

$$\frac{a^2 + b^2}{5} = \frac{(2c-d)^2 + (2d+c)^2}{5} = \frac{5(c^2 + d^2)}{5} = c^2 + d^2 = n\,.$$

Hence, such integers exist by construction.

Also solved by ARKADY ALT, San Jose, CA, USA; HOUDA ANOUN, Bordeaux, France; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; and D. KIPP JOHNSON, Beaverton, OR, USA.

M259. *Proposed by the Mayhem Staff.*

The number n is formed by concatenating the strings of digits formed by the numbers 2^{2006} and 5^{2006}. How many digits does n have?

Solution by Arkady Alt, San Jose, CA, USA.

More generally, for any natural number m, let p and q be the number of digits in the strings of digits formed by 2^m and 5^m, respectively. Then $10^{p-1} < 2^m < 10^p$ and $10^{q-1} < 5^m < 10^q$. Therefore,

$$\left(10^{p-1}\right)\left(10^{q-1}\right) \;<\; 2^m \cdot 5^m \;<\; 10^p \cdot 10^q\,;$$

that is,

$$10^{p+q-2} \;<\; 10^m \;<\; 10^{p+q}\,.$$

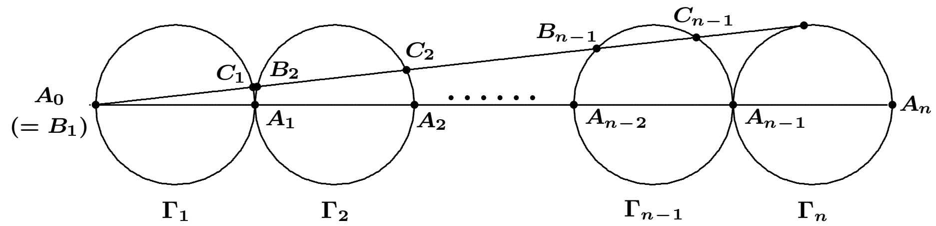
Thus, $p+q-2 < m < p+q$, which is equivalent to $m = p+q-1$, or $p+q = m+1$. We can conclude that a concatenation of 2^m and 5^m has $m+1$ digits. In particular, taking $m = 2006$, we find that n has 2007 digits.

Also solved by HOUDA ANOUN, Bordeaux, France; ALPER CAY, Uzman Private School, Kayseri, Turkey; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; D. KIPP JOHNSON, Beaverton, OR, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M260. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

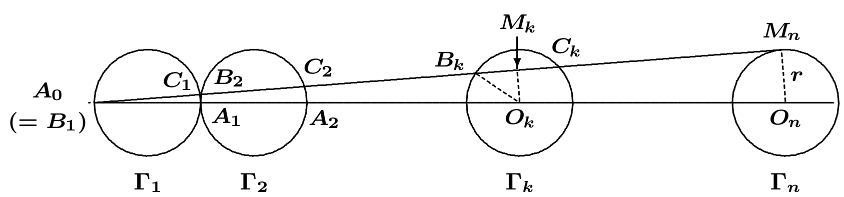
Points A_0, A_1, $\ldots$, A_n lie on a line, in that order, spaced a uniform distance $2r$ apart. For $1 \le k \le n$, let Γ_k be the circle with $A_{k-1}A_k$ as diameter. The line through A_0 tangent to Γ_n intersects the circle Γ_k at the points B_k and C_k, for $1 \le k \le n-1$.

Determine the length of the line segment B_kC_k for $1 \le k \le n-1$.



Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let O_k be the centre of Γ_k. Let M_k be the mid-point of chord B_kC_k for $1 \le k \le n-1$, and let M_n be the point of tangency to Γ_n.



Let k be fixed such that $1 \le k \le n-1$. It can be seen that triangles $A_0O_kM_k$ and $A_0O_nM_n$ are similar. We can conclude that

$$\frac{O_kM_k}{r} \;=\; \frac{A_0O_k}{A_0O_n} \;=\; \frac{2kr-r}{2nr-r} \;=\; \frac{2k-1}{2n-1}\,.$$

If we set $\theta_k = \angle B_kO_kM_k$, then $\cos\theta_k = \dfrac{O_kM_k}{r} = \dfrac{2k-1}{2n-1}$. Since $\frac{1}{2}B_kC_k = r\sin\theta_k$, we have

$$\begin{aligned} B_kC_k &= 2r\sin\theta_k \;=\; 2r\sqrt{1-\cos^2\theta_k} \\ &= 2r\sqrt{1-\left(\frac{2k-1}{2n-1}\right)^2} \;=\; \frac{2r}{2n-1}\sqrt{(2n-1)^2-(2k-1)^2} \\ &= \frac{4r}{2n-1}\sqrt{(n-k)(n+k-1)}\,. \end{aligned}$$

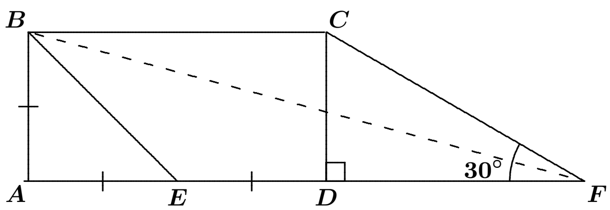
Also solved by KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were two incorrect solutions submitted.

M261. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Rectangle $ABCD$ has $AB = \frac{1}{2}BC$. On the outside of the rectangle, draw $\triangle DCF$, where $\angle DFC = 30^\circ$ and ADF is a straight line segment. Let E be the mid-point of AD.

Determine the measure of $\angle EBF$.

Essentially the same solution by ROBERT BILINSKI, Collège Montmorency, Laval, QC; ALPER CAY, Uzman Private School, Kayseri, Turkey; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.



In right triangle CDF, we have $\angle DFC = 30^\circ$ and $\angle DCF = 60^\circ$. We can then conclude that $CF = 2CD = 2AB = BC$. Now, considering isosceles triangle BCF, we have $\angle BCF = 150^\circ$ and consequently, $\angle CBF = \angle CFB = 15^\circ$. Also, we know that triangle ABE is isosceles with $\angle ABE = \angle AEB = 45^\circ$. Thus,

$$\angle EBF \;=\; 90^\circ - \angle ABE - \angle CBF \;=\; 30^\circ\,.$$

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; D. KIPP JOHNSON, Beaverton, OR, USA; and GEOFFREY A. KANDALL, Hamden, CT, USA.

M262. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(1) = 1$ and, for all real numbers x and y, we have $f(x+y) = 3^y f(x) + 2^x f(y)$.

Combination of similar solutions by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Houda Anoun, Bordeaux, France; Hasan Denker, Istanbul, Turkey; Jean-David Houle, student, McGill University, Montreal, QC; D. Kipp Johnson, Beaverton, OR, USA; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let f be any function satisfying the given conditions $f(1) = 1$ and, for all real numbers x and y,

$$f(x+y) \;=\; 3^y f(x) + 2^x f(y)\,. \tag{1}$$

Setting $y = 1$ in (1) gives, for all $x \in \mathbb{R}$,

$$f(x+1) \;=\; 3f(x) + 2^x f(1) \;=\; 3f(x) + 2^x\,. \tag{2}$$

Setting $x = 1$ in (1) gives, for all $y \in \mathbb{R}$,

$$f(1+y) \;=\; 3^y f(1) + 2f(y) \;=\; 3^y + 2f(y)\,. \tag{3}$$

Changing y to x in (3), we get, for all $x \in \mathbb{R}$,

$$f(1+x) \;=\; 3^x + 2f(x)\,. \tag{4}$$

Finally, using (2) and (4) and noting that $f(x+1) = f(1+x)$, we get

$$3f(x) + 2^x \;=\; 3^x + 2f(x)\,.$$

Thus, $f(x) = 3^x - 2^x$.

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

Problem of the Month

Ian VanderBurgh

Here is a problem that requires only some careful reasoning (albeit pretty tricky careful reasoning) and the ability to add.

Problem (2006 Grade 8 Gauss Contest)

In the diagram, the numbers from 1 to 25 are to be arranged in the 5×5 grid so that each number, except 1 and 2, is the sum of two of its neighbours. (Numbers in the grid are *neighbours* if their squares touch along a side or at a corner. For example, the "1" has 8 neighbours.) Some of the numbers have already been filled in. Which number must replace the "?" when the grid is completed?

			20	21
	6	5	4	
23	7	1	3	?
	9	8	2	
25	24			22

This is not another Sudoku—honest! It looks a bit like one, though. That is part of the reason why this problem was included on the Contest—it is nice to have problems that look familiar but, upon closer examination, are a bit different.

Solution: We could just fiddle around by trial and error until we get some numbers that work. But we will walk through the solution in a logical way.

It's tough to know exactly where to start. First, it makes sense to check which numbers are missing. The grid already includes the numbers 1 to 9 and 20 to 25; so those missing are 10 to 19.

Next, we could figure out which numbers in the grid are already the sum of two neighbours. For example, 9 has neighbours 1 and 8 (and $9 = 1 + 8$); 8 has neighbours 1 and 7 (and $8 = 1 + 7$), and so on. Let's italicize every number which is already the sum of two of its neighbours, as well as the entries 1 and 2.

			20	21
	6	*5*	*4*	
23	*7*	*1*	*3*	?
	9	*8*	*2*	
25	24			22

Now what? It's probably time for that tried and true problem-solving technique—panic. After we get that out of our system, we might try looking at some of the numbers that have almost all of their neighbours already filled in. Also, we might as well focus on the part of the grid near the "?".

For example, consider 21. Since 21 already has neighbours 20 and 4, we must write 21 as either $20 + 1$ or $4 + 17$. But the number 1 already appears elsewhere in the grid; thus, the empty space below 21 must be 17.

			20	*21*
	6	*5*	*4*	17
23	*7*	*1*	*3*	?
	9	*8*	*2*	
25	24			22

Looking at 17 as we did with 21, we see that 17 must be $3 + 14$ or $4 + 13$; thus, the "?" must represent either 13 or 14. But we can't say for sure yet which one it is.

How about 22? It cannot be $2 + 20$, as 20 is already accounted for. What two numbers add to 22 and are not yet in the grid? The only possibility is 10 and 12, in some order. But can we tell which of 10 and 12 is placed where? If 10 was above 22, we could not get 10 as the sum of two neighbours, since $2 + 8$ and $3 + 7$ are not possible. If 12 is above 22, then $12 = 10 + 2$ and $10 = 8 + 2$, which can work.

			20	*21*
	6	*5*	*4*	17
23	*7*	*1*	*3*	?
	9	*8*	*2*	*12*
25	24		*10*	*22*

We know that the "?" is either 13 or 14. Could it be 13? Are there two neighbours of "?" that add to 13? No. So the "?" must be 14, which solves the problem.

But wait! We can't stop now! Let's carry on a bit further.

Looking at 25, we see that 25 must be $24 + 1$ (not a possibility) or $9 + 16$. Hence, the number in the space above 25 must be 16. This now allows us to italicize 23, 24, 25, and 16. (Why?)

			20	*21*
	6	*5*	*4*	*17*
23	*7*	*1*	*3*	*14*
16	*9*	*8*	*2*	*12*
25	*24*		*10*	*22*

Try completing the rest of the grid on your own!