SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3164. [2006 : 394, 396] *Proposed by Mihály Bencze, Brasov, Romania.*

Let P be any point in the plane of $\triangle ABC$. Let D, E, and F denote the mid-points of BC, CA, and AB, respectively. If G is the centroid of $\triangle ABC$, prove that

$$0 \leq 3PG+PA+PB+PC-2(PD+PE+PF) \leq \tfrac{1}{2}(AB+BC+CA)\,.$$

Composite of similar solutions by Michel Bataille, Rouen, France; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The left inequality has already been proven (see the solution of problem 3052 [2006 : 341]).

As in 3052, we set $\mathrm{a} = \overrightarrow{PA}$, $\mathrm{b} = \overrightarrow{PB}$, and $\mathrm{c} = \overrightarrow{PC}$. With this notation, the right inequality can be rewritten as

$$\begin{aligned}|\mathrm{a}| + |\mathrm{b}| + |\mathrm{c}| + |\mathrm{a}+\mathrm{b}+\mathrm{c}| - |\mathrm{b}+\mathrm{c}| - |\mathrm{c}+\mathrm{a}| - |\mathrm{a}+\mathrm{b}|& \\ \leq \tfrac{1}{2}(|\mathrm{b}-\mathrm{a}| + |\mathrm{c}-\mathrm{b}| + |\mathrm{a}-\mathrm{c}|)&\,,\end{aligned}$$

or

$$\begin{aligned}&|2\mathrm{a}| + |2\mathrm{b}| + |2\mathrm{c}| + |2(\mathrm{a}+\mathrm{b}+\mathrm{c})| \\ &\qquad \leq |\mathrm{b}-\mathrm{a}| + |\mathrm{c}-\mathrm{b}| + |\mathrm{a}-\mathrm{c}| + 2(|\mathrm{b}+\mathrm{c}| + |\mathrm{c}+\mathrm{a}| + |\mathrm{a}+\mathrm{b}|)\,.\end{aligned}$$

Now, the Triangle Inequality gives us

$$\begin{aligned}|2\mathrm{a}| &= |(\mathrm{a}+\mathrm{b}) - (\mathrm{b}-\mathrm{a})| \leq |\mathrm{a}+\mathrm{b}| + |\mathrm{b}-\mathrm{a}|\,,\\ |2\mathrm{b}| &= |(\mathrm{b}+\mathrm{c}) - (\mathrm{c}-\mathrm{b})| \leq |\mathrm{b}+\mathrm{c}| + |\mathrm{c}-\mathrm{b}|\,,\\ |2\mathrm{c}| &= |(\mathrm{c}+\mathrm{a}) - (\mathrm{a}-\mathrm{c})| \leq |\mathrm{c}+\mathrm{a}| + |\mathrm{a}-\mathrm{c}|\,,\\ |2(\mathrm{a}+\mathrm{b}+\mathrm{c})| &= |(\mathrm{a}+\mathrm{b}) + (\mathrm{b}+\mathrm{c}) + (\mathrm{c}+\mathrm{a})| \\ &\leq |\mathrm{a}+\mathrm{b}| + |\mathrm{b}+\mathrm{c}| + |\mathrm{c}+\mathrm{a}|\,.\end{aligned}$$

The result follows by adding the last four inequalities.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and the proposer.

3165. [2006 : 394, 397] *Proposed by Mihály Bencze, Brasov, Romania.*

For any positive integer n, prove that there exists a polynomial $P(x)$, of degree at least $8n$, such that

$$\sum_{k=1}^{(2n+1)^2} |P(k)| \ < \ |P(0)| \,.$$

Essentially the same solution by Roy Barbara, Lebanese University, Fanar, Lebanon; Michel Bataille, Rouen, France; Richard I. Hess, Rancho Palos Verdes, CA, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA.

Let n be a positive integer. Let $P(x) = \prod\limits_{k=1}^{(2n+1)^2} (x-k)$. Then $P(k) = 0$ for $k = 1, 2, \ldots, (2n+1)^2$, and therefore,

$$\sum_{k=1}^{(2n+1)^2} |P(k)| \ = \ 0 \ < \ (2n+1)^2! \ = \ |P(0)| \,.$$

The degree of $P(x)$ is $(2n+1)^2 \geq 8n$ (note that this inequality is equivalent to $(2n-1)^2 \geq 0$).

Also solved by M.R. MODAK, Pune, India; and the proposer.

The solution by Modak was the same as the one above except that he defined $P(x)$ as the product of $x-k$ for $k=2$ to $k=(2n+1)^2$ instead of $k=1$ to $k=(2n+1)^2$. Thus, his polynomial $P(x)$ has degree $(2n+1)^2 - 1 = 4n(n+1)$. The proposer's solution was considerably more complicated, involving Chebyshev polynomials.

3166. [2004–118] *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

Let P be an interior point of the triangle ABC. Denote by d_a, d_b, d_c the distances from P to the sides BC, CA, AB, respectively, and denote by D_A, D_B, D_C the distances from P to the vertices A, B, C, respectively. Further let P_A, P_B, and P_C denote the measures of $\angle BPC$, $\angle CPA$, and $\angle APB$, respectively.

Prove that

$$\begin{aligned} d_a d_b \sin\left(\tfrac{1}{2}(P_A+P_B)\right) + d_b d_c \sin\left(\tfrac{1}{2}(P_B+P_C)\right) + d_c d_a \sin\left(\tfrac{1}{2}(P_C+P_A)\right) \\ \leq \ \tfrac{1}{4}(D_B D_C \sin P_A + D_C D_A \sin P_B + D_A D_B \sin P_C)\,. \end{aligned}$$

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $[XYZ]$ represent the area of $\triangle XYZ$. Then the right side of the given inequality is simply $\frac{1}{2}[ABC]$.

Let the interior angle bisectors of $\angle BPC$, $\angle CPA$, and $\angle APB$ meet the sides BC, CA, and AB at A', B', and C', respectively. Then $PA' \geq d_a$, $PB' \geq d_b$, and $PC' \geq d_c$. Thus, the left side of the given inequality is less than twice the sum of $[A'PB']$, $[B'PC']$, and $[C'PA']$; that is, the left side of the given inequality is less than $2[A'B'C']$.

Therefore, it suffices to prove that $[A'B'C'] \leq \frac{1}{4}[ABC]$.

But $A'B'C'$ is a *Cevian triangle* of $\triangle ABC$; that is, AA', BB', and CC' are concurrent. This follows since

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} \;=\; \frac{PB}{PC} \cdot \frac{PC}{PA} \cdot \frac{PA}{PB} \;=\; 1\,.$$

The desired result then follows from the following theorem:

Theorem. Let AA', BB', and CC' be three concurrent Cevians of $\triangle ABC$. Then $[A'B'C'] \leq \frac{1}{4}[ABC]$.

Proof: Let

$$\lambda \;=\; \frac{AC'}{C'B}\,, \quad \mu \;=\; \frac{BA'}{A'C}\,, \quad \text{and} \quad \nu \;=\; \frac{CB'}{B'A}\,.$$

From Ceva's Theorem, we have $\lambda\mu\nu = 1$. Then

$$[BA'C'] \;=\; \frac{\mu[ABC]}{(1+\mu)(1+\lambda)}\,, \quad [CB'A'] \;=\; \frac{\nu[ABC]}{(1+\nu)(1+\mu)}\,,$$
$$\text{and} \quad [AC'B'] \;=\; \frac{\lambda[ABC]}{(1+\lambda)(1+\nu)}\,.$$

Hence,

$$\begin{aligned}
\frac{[A'B'C']}{[ABC]} &= 1 - \frac{\mu}{(1+\mu)(1+\lambda)} - \frac{\nu}{(1+\nu)(1+\mu)} - \frac{\lambda}{(1+\lambda)(1+\nu)} \\
&= 1 - \frac{\mu(1+\nu)+\nu(1+\lambda)+\lambda(1+\mu)}{(1+\lambda)(1+\mu)(1+\nu)} \\
&= 1 - \frac{\lambda+\mu+\nu+\lambda\mu+\mu\nu+\nu\lambda}{(1+\lambda)(1+\mu)(1+\nu)} \\
&= \frac{1+\lambda\mu\nu}{1+\lambda+\mu+\nu+\lambda\mu+\mu\nu+\nu\lambda+\lambda\mu\nu} \\
&= \frac{2}{2+(\lambda+1/\lambda)+(\mu+1/\mu)+(\nu+1/\nu)} \qquad \text{since } \lambda\mu\nu = 1
\end{aligned}$$

which is obviously less than or equal to $\frac{2}{8} = \frac{1}{4}$ because each of the bracketed expressions is at least 2. ■

Also solved by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3167. [2006 : 395, 397] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let ABC be a non-obtuse triangle with circumradius R. If a, b, c are the lengths of the sides opposite angles A, B, C, respectively, prove that

$$a\cos^3 A + b\cos^3 B + c\cos^3 C \;\le\; \frac{abc}{4R^2}\,.$$

Composite of similar solutions by Mohammed Aassila, Strasbourg, France; and Vedula N. Murty, Dover, PA, USA.

Let S be the area of triangle ABC. Since

$$S \;=\; \frac{abc}{4R} \;=\; \frac{1}{2}R^2\sum_{\text{cyclic}}\sin 2A\,,$$

$$\frac{a}{\sin A} \;=\; \frac{b}{\sin B} \;=\; \frac{c}{\sin C} \;=\; 2R\,,$$

and $\displaystyle\sum_{\text{cyclic}}\sin 4A \;=\; -4\sin 2A\sin 2B\sin 2C\,,$

we have

$$\begin{aligned}\sum_{\text{cyclic}} a\cos^3 A \;&=\; \sum_{\text{cyclic}}(2R\sin A)\cos^3 A \;=\; R\sum_{\text{cyclic}}\sin 2A\cos^2 A\\ &=\; \frac{1}{2}R\sum_{\text{cyclic}}\sin 2A(1+\cos 2A)\\ &=\; \frac{1}{2}R\sum_{\text{cyclic}}\sin 2A + \frac{1}{4}R\sum_{\text{cyclic}}\sin 4A\\ &=\; \frac{abc}{4R^2} - R\sin 2A\sin 2B\sin 2C\,.\end{aligned}$$

Now we note that $\sin 2A\sin 2B\sin 2C \ge 0$ because $\triangle ABC$ is non-obtuse. Thus, we obtain the desired inequality.

Equality holds if and only if the triangle is right-angled.

Also solved by MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Howard observed that

$$\sum_{\text{cyclic}} a\cos^3 A \;>\; \frac{abc}{4R^2}$$

if and only if the triangle is obtuse, a fact that follows easily from the featured solution as well.

3168. [2006 : 395, 397] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let x_1, x_2, $\ldots$, x_n be positive real numbers satisfying $\prod_{i=1}^{n} x_i = 1$. Prove that
$$\sum_{i=1}^{n} x_i^n(1+x_i) \;\geq\; \frac{n}{2^{n-1}} \prod_{i=1}^{n}(1+x_i)\,.$$

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

By the Power–Mean Inequality, we have $\sqrt[n]{\frac{1}{2}(a^n+1^n)} \geq \frac{1}{2}(a+1)$; this is equivalent to $a^n+1 \geq \frac{1}{2^{n-1}}(a+1)^n$. Using this result as well as the AM–GM Inequality and the given condition $x_1x_2\cdots x_n = 1$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} x_i^n(1+x_i) &= x_1^n+\cdots+x_n^n+x_1^{n+1}+\cdots+x_n^{n+1} \\
&\geq x_1^n+\cdots+x_n^n+n\sqrt[n]{(x_1x_2\cdots x_n)^{n+1}} \\
&= x_1^n+\cdots+x_n^n+n \;=\; (x_1^n+1)+\cdots+(x_n^n+1) \\
&\geq \frac{1}{2^{n-1}}\left((x_1+1)^n+\cdots+(x_n+1)^n\right) \\
&\geq \frac{n}{2^{n-1}}\sqrt[n]{(x_1+1)^n(x_2+1)^n\cdots(x_n+1)^n} \\
&= \frac{n}{2^{n-1}}\prod_{i=1}^{n}(1+x_i)\,.
\end{aligned}$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Cao Minh remarked that the case $n = 3$ is problem 11 (Russia) of IMO Short List 1998.

3169. [2006 : 395, 397] *Proposed by Vesselin Dimitrov, National High-school of Mathematics and Science, Sofia, Bulgaria.*

Let A be a finite set of real numbers such that each $a \in A$ is uniquely expressible as $a = b + c$, where b, $c \in A$ and $b \leq c$.

(a) Prove that there exist distinct elements a_1, a_2, $\ldots$, $a_k \in A$ such that $a_1 + a_2 + \cdots + a_k = 0$.

(b)★ Does this necessarily hold if it is no longer assumed that each representation $a = b + c$ is unique?

No correct solutions were received for either part (a) or (b), so this problem remains open.

The proposer remarked that there are many finite sets $A \subset \mathbb{Z}$ for which the given condition holds. He claims, for example, that for each $n \in \mathbb{N}$, the set $\{ -2^{n+1} + 2^k + 1,\, 2^k \mid k = 0, 1, 2, \dots, n\}$ satisfies the requirement. [*Ed.*: Note that

$$\begin{aligned} 2^k &= 2^{k-1} + 2^{k-1} \qquad \text{if } k \geq 1\,, \\ 2^0 = 1 &= \left(-2^{n+1} + 2^n + 1\right) + 2^n\,, \\ -2^{n+1} + 2^k + 1 &= \left(-2^{n+1} + 2^{k-1} + 1\right) + 2^{k-1} \quad \text{if } k \geq 1\,, \\ -2^{n+1} + 2^0 + 1 &= -2^{n+1} + 2 \;=\; 2\left(-2^{n+1} + 2^n + 1\right). \end{aligned}$$

It is not difficult to verify that each of these representations is unique.]

3170. [2006 : 395, 398] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a and b be real numbers satisfying $0 \leq a \leq \frac{1}{2} \leq b \leq 1$. Prove that

(a) $2(b-a) \leq \cos \pi a - \cos \pi b$;

(b) $(1-2a)\cos \pi b \leq (1-2b)\cos \pi a$.

Solution by Michel Bataille, Rouen, France.

(a) Let $f(x) = 2x + \cos \pi x$. The proposed inequality can then be expressed as $f(b) \leq f(a)$.

We have $f'(x) = 2 - \pi \sin \pi x$ and $f''(x) = -\pi^2 \cos \pi x$. Hence, f' is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$. Since $f'(0) = f'(1) = 2$ and $f'\left(\frac{1}{2}\right) = 2 - \pi < 0$, there exist α and β with $0 < \alpha < \frac{1}{2} < \beta < 1$, such that $f'(\alpha) = f'(\beta) = 0$ and $f'(x) < 0$ if and only if $x \in (\alpha, \beta)$.

Thus, f is increasing on $[0, \alpha]$ and $[\beta, 1]$, and decreasing on $[\alpha, \beta]$. Since $f(0) = f\left(\frac{1}{2}\right) = f(1) = 1$, we see that $f(x) \geq 1$ for $x \in \left[0, \frac{1}{2}\right]$ and $f(x) \leq 1$ for $x \in \left[\frac{1}{2}, 1\right]$. In particular, $f(b) \leq 1 \leq f(a)$, and the result follows.

(b) (Modified slightly by the editor). The proposed inequality is false; for example, if $a = \frac{1}{4}$ and $b = 1$, then the inequality would imply that $-\frac{1}{2} \leq -\frac{\sqrt{2}}{2}$, which is absurd.

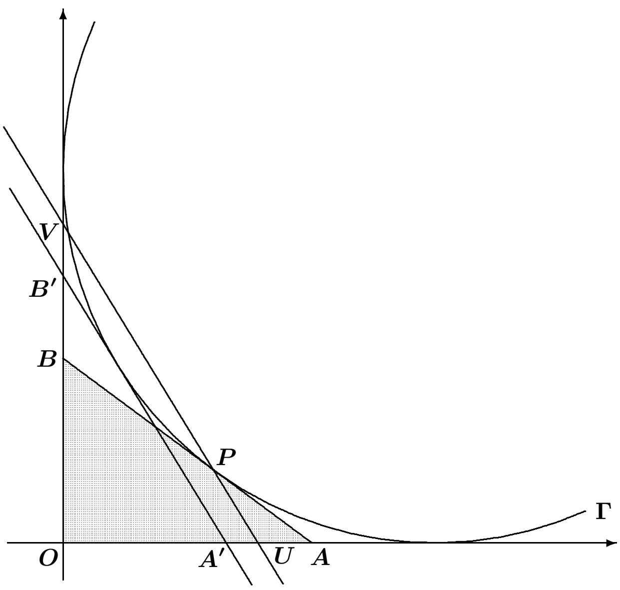
Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer (part (a) *only).*

All the solvers noticed that the inequality in (b) *is incorrect. Curtis commented that the inequality does hold sometimes (for example, when $a = 0$ and $b = \frac{3}{4}$); thus, one cannot simply reverse it.*

3171. [2006 : 395, 398] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a point P in the first quadrant, it is known that the line segment in the first quadrant joining the coordinate axes, passing through P, and having minimum length (Philo's line) is not constructible using straightedge and compass. However, the line which (together with the two axes) defines a triangle in the first quadrant with minimum perimeter is constructible. Give such a construction.

I. *Solution by Claudio Arconcher, Jundiaí, Brazil.*



Claim. The hypotenuse of the triangle of minimum perimeter is the tangent at P to the circle through P, call it Γ, that is tangent to the positive x- and y-axes and separated by that tangent from the origin O.

Proof: Let the line through P tangent to Γ meet the x-axis at A and the y-axis at B. Let ℓ be any other line through P intersecting the positive axes at points U and V, say. Then the line parallel to UV and tangent to Γ intersects the axes at A' and B' with $OA' < OU$ and $OB' < OV$. Since $OA + OB + AB = OA' + OB' + A'B'$, which equals twice the length of the tangents to Γ from O, we have

$$OA + OB + AB \;=\; OA' + OB' + A'B' \;<\; OU + OV + UV\,,$$

as claimed. ∎

Construction. First construct an arbitrary circle Γ' that is tangent to the positive x- and y-axes (whose centre C' is an arbitrary point of the line $y = x$), and call P' the point closest to O where OP intersects Γ'. Define C to be the point where the line parallel to $P'C'$ through P intersects OC'. Then C is the centre of Γ (because the dilatation with centre O that takes C' to C will take P' to P, and take Γ' and its points of tangency with the axes to Γ and its tangency points on the axes).

II. *Composite of similar solutions by Peter Y. Woo, Biola University, La Mirada, CA, USA; and the proposer.*

Analysis. Let the axes meet at O, and let the line segment through $P(a, b)$ meet the x-axis at A and the y-axis at B. Define $\theta = \angle BAO$. Without loss of generality assume that $a \geq b$. Then the perimeter of triangle OAB is

$$p(\theta) \;=\; a(1 + \sec\theta + \tan\theta) + b(1 + \csc\theta + \cot\theta)\,.$$

Its derivative satisfies

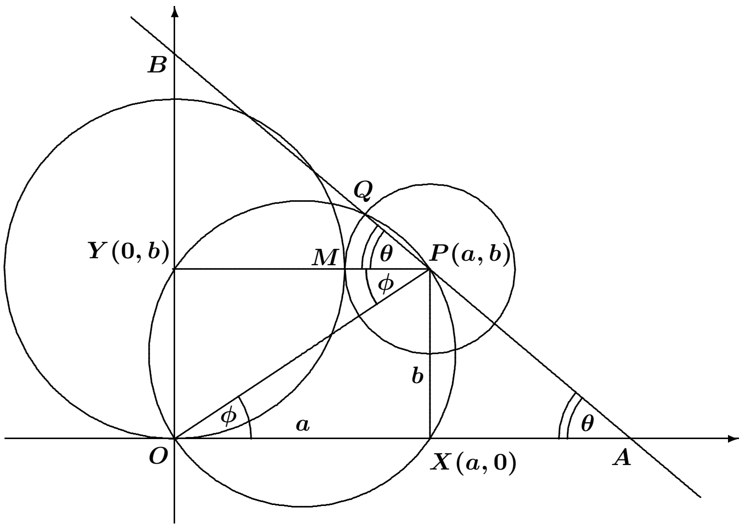
$$p'(\theta) \;=\; \frac{a}{1 - \sin\theta} - \frac{b}{1 - \cos\theta}\,.$$

The geometry indicates that the minimum perimeter occurs when the derivative is zero, which means $a\cos\theta - b\sin\theta = a - b$, or

$$\frac{a\cos\theta - b\sin\theta}{\sqrt{a^2 + b^2}} \;=\; \frac{a - b}{\sqrt{a^2 + b^2}}\,.$$

With $\phi = \angle AOP$, so that $\cos\phi = \dfrac{a}{\sqrt{a^2 + b^2}}$ and $\sin\phi = \dfrac{b}{\sqrt{a^2 + b^2}}$, the last equation can be interpreted as

$$\cos(\phi + \theta) \;=\; \frac{a - b}{\sqrt{a^2 + b^2}}\,.$$



Construction. Construct the circle with diameter OP, cutting the y-axis again at $Y(0,b)$. Construct the circle with centre Y and radius YO, cutting the segment YP at M. Draw the circle with centre P and radius PM, cutting the first circle at Q (between P and B). Then PQ is the desired line that hits the axes at A and B and determines the triangle OAB of minimum perimeter. (Since $PQ = a - b$ and $PQ \perp OQ$, we get $\cos\angle QPO = \frac{a-b}{\sqrt{a^2+b^2}}$; whence, the line PQ makes an angle of θ with the x-axis such that $\theta + \phi = \angle OPQ$.)

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; and CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA.

3172. [2006 : 396, 398] *Proposed by Vincenţiu Rădulescu, University of Craiova, Craiova, Romania.*

Let f be a positive continuous function defined on $(0,\infty)$ such that $\liminf\limits_{x\to\infty} f(x) > 0$. Prove that there is no positive, twice differentiable function g defined on $[0,\infty)$ which satisfies $g'' + f\circ g = 0$.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA, modified by the editor.

Suppose that such a function g exists. Then $f\circ g$ is continuous, and therefore g'' is also continuous. Since f is positive on $(0,\infty)$, the function g'' is negative on $(0,\infty)$, and g' is decreasing.

Now suppose that $g'(\alpha) = -k < 0$ for some $\alpha > 0$. Then $g'(x) \le -k$ for all $x \ge \alpha$. Therefore, $g(x) \le g(\alpha) - k(x-\alpha)$ for all $x \ge \alpha$, implying that g is eventually negative, a contradiction. Hence, $g'(x) \ge 0$ on $[0,\infty)$ and g is increasing.

Since g is positive and increasing, $\lim\limits_{x\to\infty} g(x) = \infty$ or $\lim\limits_{x\to\infty} g(x) = M$ for some positive real number M. If $\lim\limits_{x\to\infty} g(x) = \infty$, then

$$\liminf_{x\to\infty} f\bigl(g(x)\bigr) \;=\; \liminf_{y\to\infty} f(y) \;>\; 0\,.$$

If $\lim\limits_{x\to\infty} g(x) = M$, then, using the continuity of f, we have

$$\liminf_{x\to\infty} f\bigl(g(x)\bigr) \;=\; \lim_{y\to M} f(y) \;=\; f(M) \;>\; 0\,.$$

Thus, in both cases, $\liminf\limits_{x\to\infty} f\bigl(g(x)\bigr) > 0$.

Now, since $g'' = -f\circ g$, we have

$$\limsup_{x\to\infty} g''(x) \;=\; -\liminf_{x\to\infty} f\bigl(g(x)\bigr) \;<\; 0\,.$$

Therefore, there exist $\delta > 0$ and $\beta > 0$ such that $g''(x) \le -\delta$ for all $x \ge \beta$. Then $g'(x) \le g'(\beta) - \delta(x-\beta)$ for all $x \ge \beta$, implying that g' is eventually negative, a contradiction.

Hence, such a function g does not exist.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; KEE-WAI LAU, Hong Kong, China; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer

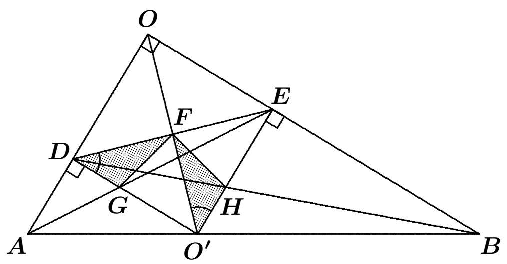
WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, commented that this problem is Aufgabe 1224 of the Swiss journal Elemente der Matematik, *and that a solution can be found in the "Aufgaben" section of issue No. 4 of Vol. 61 (2006).*

3173. [2006 : 396, 398] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let OAB be a right triangle with right angle at O. Let OO' be the bisector of angle O, with O' on AB. Let D and E be the feet of the perpendiculars from O' to the legs OA and OB, respectively. Let $F = OO' \cap DE$, $G = AE \cap O'D$, and $H = BD \cap O'E$.

Prove that $\triangle FGH$ is an isosceles right triangle with right angle at F.

Composite of similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.



Since O' lies on the internal bisector of $\angle AOB$, we have $DO' = O'E$, and therefore, the rectangle $ODO'E$ is a square. Hence, $FD = FO'$ and $\angle FDG = \angle FO'H = 45^\circ$. Since $\triangle DAG$ and $\triangle OAE$ are similar, as are $\triangle HO'B$ and $\triangle DAB$, we obtain

$$\frac{DG}{AD} = \frac{OE}{AO} = \frac{OD}{AO} = \frac{O'B}{AB} = \frac{O'H}{AD}.$$

Hence, $DG = O'H$. It follows that $\triangle FDG$ and $\triangle FO'H$ are congruent. Thus, $FG = FH$ and $\angle DFG = \angle O'FH$, which implies that $\angle GFH = 90^\circ$ (because $\angle DFO' = 90^\circ$). Consequently, $\triangle FGH$ is isosceles with a right angle at F.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3174. [2006 : 396, 398] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given $\triangle ABC$, we define A' to be the point where the internal angle bisector of angle A meets the side BC. Let B' and C' be the feet of the perpendiculars from A' to the sides AC and AB, respectively. Prove that BB' and CC' intersect on the altitude from A.

Composite of similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

We will use directed distances in Ceva's Theorem to avoid any need for special cases. From the congruent right triangles $AC'A'$ and $AB'A'$, we deduce that $AC' = AB'$; that is,

$$\frac{AC'}{B'A} = 1\,.$$

Let D be the foot of the altitude from A. From the similar right triangles $C'BA'$ and DBA, we have

$$\frac{BD}{C'B} = \frac{AB}{BA'}\,,$$

and from the similar right triangles $A'B'C$ and ADC, we have

$$\frac{CB'}{DC} = \frac{A'C}{AC}\,.$$

Multiplying together these three equations, we obtain

$$\frac{AC'}{B'A}\cdot\frac{BD}{C'B}\cdot\frac{CB'}{DC} = \frac{AB}{AC}\cdot\frac{A'C}{BA'}\,.$$

Since A' lies on the bisector of angle A, we see that A' divides the segment BC in the ratio $AB : AC$; whence, $\frac{BA'}{A'C} = \frac{AB}{AC}$; that is,

$$\frac{AB}{AC}\cdot\frac{A'C}{BA'} = 1\,.$$

We conclude that

$$\frac{AC'}{C'B}\cdot\frac{BD}{DC}\cdot\frac{CB'}{B'A} = 1\,,$$

and the desired result follows from the converse of Ceva's Theorem. [Almost! See the remarks following the list of solvers.]

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; MICHAEL PARMENTER, Memorial University of Newfoundland, *St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

Most solvers used some variant of the featured solution, but only Bataille and Parmenter noted that the converse of Ceva's Theorem asserts that the lines AD, BB', and CC' are **parallel or concurrent**. *Here is how Bataille completed the argument: $\angle A'B'C = 90°$; hence, the foot B'' of the perpendicular from B' to BC lies between A' and C; since A' lies between B and C, it follows that $B'' \neq B$, and the line $B'B$ is not parallel to AD. Consequently, AD, BB', CC' must be concurrent.*

3175. [2006 : 396, 398] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let $\triangle ABC$ be a triangle with $\angle B > 90°$ and $\angle A < 60°$. Let P be a point on the side AB such that $\angle CPB = 60°$. Let D be the point on CP which also lies on the interior angle bisector of $\angle A$. If $\angle CBD = 30°$, prove that CP is a trisector of angle ACB.

I. *Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.*

Let AY be the bisector of $\angle CAB$, let $PM \perp AB$ with M on AC, let CN be the bisector of $\angle ACP$, let F be the point on AM with $\angle FPA = 60°$, let H be the point of intersection of CN and PM, and let E be the point of intersection of CN and AD. Denote the angles of $\triangle ABC$ by α, β, and γ as usual.

It is clear that $\angle FPM = \angle MPC = 30°$. From $\triangle PAC$, we obtain $\alpha + \angle ACP = 60°$; then $\angle ACN = \angle NCP = \frac{1}{2}\angle ACP = 30° - \frac{1}{2}\alpha$. From $\triangle APD$, we get $\frac{1}{2}\alpha + \angle ADP = 60°$; then $\angle ADP = 60° - \frac{1}{2}\alpha$.

In $\triangle FPC$, the line PM is the bisector of $\angle FPC$ and the line CN is the bisector of $\angle ACP$. Thus, H is the incentre of $\triangle FPC$. Hence, the line FH is the bisector of $\angle MFP$. From $\triangle APF$, we see that $\angle PFM = \alpha + 60°$, so that $\angle MFH = \angle HFP = \frac{1}{2}\alpha + 30°$.

From $\triangle FHC$, we obtain

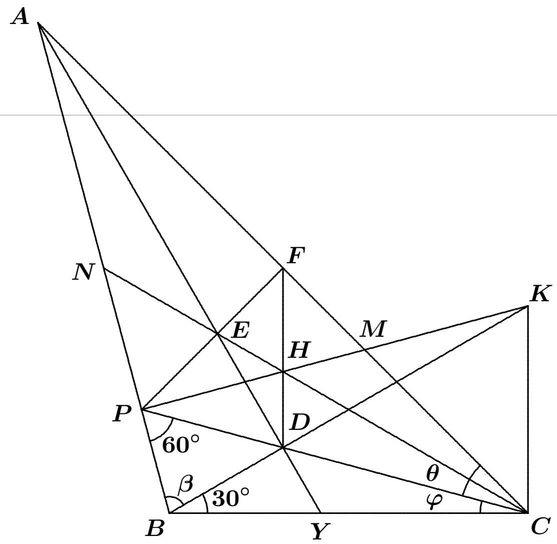
$$\begin{aligned}\angle FHE &= \angle FCH + \angle CFH \\ &= \left(30° - \tfrac{1}{2}\alpha\right) + \left(30° + \tfrac{1}{2}\alpha\right) \;=\; 60°\,.\end{aligned}$$

Thus, in $\triangle FHP$, we have

$$\begin{aligned}\angle EHP &= 180° - \angle HPF - \angle HFP - \angle EHF \\ &= 180° - 30° - \left(\tfrac{1}{2}\alpha + 30°\right) - 60° \;=\; 60° - \tfrac{1}{2}\alpha\,.\end{aligned}$$

Therefore, $\angle EDP = \angle ADP = 60° - \frac{1}{2}\alpha = \angle EHP$, which implies that quadrilateral $EHDP$ is cyclic. Thus, $\angle CHD = \angle EPD = 60°$, and hence, the points F, H, and D are collinear.

Let PM intersect BD at K. If $\angle DBC = 30°$, then quadrilateral $PBCK$ is cyclic, since $\angle KPC = \angle KBC = 30°$. Thus, $\angle DCB = \angle PKB$ and $\angle CKB = \angle CPB = 60°$. From above, we have $\angle CHD = 60°$; whence, $\angle CKB = \angle CHD$. Therefore, quadrilateral $KHDC$ is cyclic, which implies that $\angle HKD = \angle HCD$. It follows that $\angle DCB = \angle HCD = \angle ACH$.



II. *Solution by Geoffrey A. Kandall, Hamden, CT, USA.*

Set $\theta = \angle ACD$, $\varphi = \angle DCB$, and $\beta = \angle PBD$. From $\triangle PBC$, we see that $\beta + \varphi = 90^\circ$.

From the trigonometric form of Ceva's Theorem, we have

$$\frac{\sin\angle BAD}{\sin\angle DAC}\cdot\frac{\sin\theta}{\sin\varphi}\cdot\frac{\sin 30^\circ}{\sin\beta} \;=\; 1\,.$$

Hence,

$$\sin\theta \;=\; 2\sin\varphi\sin\beta \;=\; 2\sin\varphi\cos\varphi \;=\; \sin 2\varphi\,.$$

Since the angles θ and 2φ are not supplementary ($\theta < 60^\circ$ and $\varphi < 30^\circ$), we conclude that $\theta = 2\varphi$.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; M.R. MODAK, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3177. [2006 : 462, 464] *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

Let P be any interior point of triangle $A_1A_2A_3$. Let T_1, T_2, T_3 denote the projections of P onto the sides A_2A_3, A_3A_1, A_1A_2, respectively, and let H_1, H_2, H_3 denote the orthocentres of triangles $A_1T_2T_3$, $A_2T_3T_1$, $A_3T_1T_2$, respectively. Prove that the lines H_1T_1, H_2T_2, H_3T_3 are concurrent.

A composite of similar solutions by Apostolis K. Demis, Varvakeio High School, Athens, Greece; and Taichi Maekawa, Takatsuki City, Osaka, Japan.

Because the lines T_2H_1 and PT_3 are both perpendicular to A_1A_2, they are parallel. Likewise, $T_3H_1 \parallel PT_2$; whence $H_1T_2PT_3$ is a parallelogram. In the same way, $H_2T_3PT_1$ is a parallelogram. Consequently, H_1T_2 is parallel and equal to its opposite side T_3P, which is parallel and equal to its opposite side H_2T_1. It follows that $H_1H_2T_1T_2$ is a parallelogram, so that

diagonals H_1T_1 and H_2T_2 have a common mid-point.

Similarly, using parallelograms $H_2T_3PT_1$ and $H_3T_1PT_2$, we deduce that $H_2H_3T_2T_3$ is a parallelogram; whence

diagonals H_2T_2 and H_3T_3 have a common mid-point.

Consequently, the segments H_1T_1, H_2T_2, and H_3T_3 have a common mid-point—the lines they determine are concurrent, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain (2 solutions); MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

In their second solution Bellot Rosado and López Chamorro determine that, with $A_1A_2A_3$ as the triangle of reference, if P has trilinear coordinates (p,q,r), then the common point of H_1T_1, H_2T_2, and H_3T_3 has coordinates

$$(p + r\cos B + q\cos C,\ q + p\cos C + r\cos A,\ r + q\cos A + p\cos B)\,.$$

If instead you let (p,q,r) be the areal coordinates of P, then Bataille shows that the common point has areal coordinates

$$(1-p, 1-q, 1-r)\,,$$

from which he deduces that the centroid of $\triangle T_1T_2T_3$ lies two-thirds of the way from P to the common point.