THE OLYMPIAD CORNER

No. 264

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We begin this number of the *Corner* with three problems from the 2003 Kürschák Competition in Hungary. Thanks go to Christopher Small, Canadian Team leader to the IMO in Athens, for collecting them.

2003 KÜRSCHÁK COMPETITION

1. Let EF be a diameter of the circle Γ, and let e be the tangent line to Γ at E. Let A and B be any two points of e such that E is an interior point of the segment AB, and $AE \cdot EB$ is a fixed constant. Let AF and BF meet Γ at A' and B', respectively. Prove that all such segments $A'B'$ pass through a common point.

2. We define a k-colouring of a graph G to be a colouring of its vertices using k possible colours such that the end-points of any edge have different colours. We say that G is uniquely k-colourable if G has a k-colouring and any two vertices which have the same colour in one k-colouring have the same colour in every k-colouring. Prove that if a graph G with n vertices ($n \geq 3$) is uniquely 3-colourable, then the number of its edges is at least $2n - 3$.

3. Prove that the following inequality holds for all positive integers n with the exception of finitely many n:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \gcd(i,j) \;>\; 4n^2 \,.$$

Next we give the Seniors Level problems from the 21st Hellenic Mathematical Olympiad "Archimedes" given February 7, 2004. Thanks again go to Christopher Small.

HELLENIC MATHEMATICAL COMPETITIONS 2004
Seniors Level

1. Find the greatest possible value of the positive real number M such that, for all x, y, $z \in \mathbb{R}$,

$$x^4 + y^4 + z^4 + xyz(x+y+z) \;\geq\; M(xy+yz+zx)^2 \,.$$

2. Prove that there do not exist positive integers $x_1, x_2, \ldots, x_m$, where $m \geq 2$, such that $x_1 < x_2 < \cdots < x_m$ and $\sum\limits_{i=1}^{m} x_i^{-3} = 1$.

3. A circle (O, r) and a point A outside the circle are given. From A we draw a straight line ε, different from the line AO, which intersects the circle at B and Γ, with B between A and Γ. Next we draw the symmetric line of ε with respect to the axis AO, which intersects the circle at E and Δ, with E between A and Δ.

Prove that the diagonals of the quadrilateral $B\Gamma\Delta E$ pass through a fixed point; that is, they always intersect at the same point, independent of the position of the line ε.

4. Let M be a subset of the natural numbers with 2004 elements. If we know that there is no element in M which is equal to the sum of any two other elements of M, determine the minimum value of the greatest element of M.

Next are the 10 problems of the Vietnamese Mathematical Olympiad in 2004. Thanks go to Christopher Small for collecting them.

VIETNAMESE MATHEMATICAL OLYMPIAD 2004

1. Solve the system of equations

$$\begin{aligned} x^3 + x(y-z)^2 &= 2\,, \\ y^3 + y(z-x)^2 &= 30\,, \\ z^3 + z(x-y)^2 &= 16\,. \end{aligned}$$

2. Solve the system of equations

$$\begin{aligned} x^3 + 3xy^2 &= -49\,, \\ x^2 - 8xy + y^2 &= 8y - 17x\,. \end{aligned}$$

3. Let ABC be a triangle in a plane. The internal angle bisector of $\angle ACB$ cuts the side AB at D.

Consider an arbitrary circle Γ_1 passing through C and D so that the lines BC and CA are not its tangents. This circle cuts the lines BC and CA again at M and N, respectively.

(a) Prove that there exists a circle Γ_2 touching the line DM at M and touching the line DN at N.

(b) The circle Γ_2 from part (a) cuts the lines BC and CA again at P and Q, respectively. Prove that the measures of the segments MP and NQ are constant as Γ_1 varies.

4. Given an acute triangle ABC inscribed in a circle Γ in a plane, let H be its orthocentre. On the arc BC of Γ not containing A, take a point P distinct from B and C. Let D be the point such that $\overrightarrow{AD} = \overrightarrow{PC}$. Let K be the orthocentre of triangle ACD, and let E and F be the orthogonal projections of K onto the lines BC and AB, respectively. Prove that the line EF passes through the mid-point of HK.

5. Consider the sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ defined by $x_1 = 1$ and

$$x_{n+1} = \frac{(2+\cos 2\alpha)x_n + \cos^2\alpha}{(2-2\cos 2\alpha)x_n + 2 - \cos 2\alpha}$$

for every $n = 1, 2, \ldots$, where α is a real parameter. For each $n = 1, 2, \ldots$, let $y_n = \sum\limits_{k=1}^{n} \frac{1}{2x_k+1}$. Determine all values of α so that the sequence $\{y_n\}_{n=1}^{\infty}$ has a finite limit. Find this limit in these cases.

6. Find the least value and the greatest value of the expression

$$P = \frac{x^4+y^4+z^4}{(x+y+z)^4},$$

where x, y, and z are positive real numbers satisfying the condition

$$(x+y+z)^3 = 32xyz.$$

7. Find all triples of positive integers (x, y, z) satisfying the condition

$$(x+y)(1+xy) = 2^z.$$

8. Let A be the set of the first 16 positive integers. Find the least positive integer k satisfying the following condition: in each subset consisting of k elements of A, there exist two distinct elements a and b such that a^2+b^2 is a prime number.

9. Let n be an integer, $n \geq 2$. Prove that for every integer k such that $2n-3 \leq k \leq n(n-1)/2$, there exist n distinct real numbers $a_1, a_2, \ldots, a_n$ such that among all numbers of the form $a_i + a_j$, $1 \leq i < j \leq n$, there exist exactly k distinct numbers.

10. For every positive integer n, let $S(n)$ be the sum of all digits in the decimal representation of n. If m is a positive integral multiple of 2003, find the least value of $S(m)$.

And to round out your problem pleasures, we give the Selected Camp Problems from the 2004 Taiwanese Mathematical Olympiad. Once again, thanks go to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them for our use.

2004 TAIWANESE MATHEMATICAL OLYMPIAD
Selected Camp Problems

1. Let $\mathbb{N}_0$ denote the set of non-negative integers. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(3m + 2n) = f(m) \cdot f(n)$ for all m, $n \in \mathbb{N}_0$.

2. Find all pairs of positive integers (a, b) satisfying

$$\sqrt{\frac{ab}{2b^2 - a}} = \frac{a + 2b}{4b}\,.$$

3. Suppose that the points D and E lie on the circumcircle of $\triangle ABC$, ray $\overrightarrow{AD}$ is the interior angle bisector of $\angle BAC$, and ray $\overrightarrow{AE}$ is the exterior angle bisector of $\angle BAC$. Let F be the symmetrical point of A with respect to D, and let G be the symmetrical point of A with respect to E. Prove that, if the circumcircle of $\triangle ADG$ and the circumcircle of $\triangle AEF$ intersect at P, then AP is parallel to BC.

4. Let O and H be the circumcentre and orthocentre of an acute triangle ABC. Suppose that the bisectrix of $\angle BAC$ intersects the circumcircle of $\triangle ABC$ at D, and that the points E and F are symmetrical points of D with respect to BC and O, respectively. If AE and FH intersect at G and if M is the mid-point of BC, prove that GM is perpendicular to AF.

5. A one-to-one function $f : \mathbb{Z} \to \mathbb{R}$ is given (where $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers). Also given are n different positive integers $a_1, a_2, \ldots, a_n$. Prove that there exists an integer p such that, among the set of $2n$ integers $p - a_1$, $p + a_1$, $p - a_2$, $p + a_2$, $\ldots$, $p - a_n$, $p + a_n$, there are at least n integers b such that $f(b) \geq f(p)$.

6. The seats at the Christmas Feast for the company "Enough" are arranged in a square consisting of 10 rows with 10 seats in each row. All 100 workers have different salaries. Each of them asks all his neighbours (those workers sitting immediately beside him, in front of him, or behind him—four people at most) how much they earn. A worker feels content with his salary only if he has at most one neighbour who earns more than himself. What is the maximum possible number of workers who are satisfied with their salaries?

Next we give an alternate solution to a problem of the Singapore Mathematical Olympiad given in [2005 : 216]. A solution was published last year [2006 : 386].

4. Find all real-valued functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$ defined on the set of all rational numbers $\mathbb{Q}$ satisfying the conditions

$$f(x+y) = f(x) + f(y) + 2xy ,$$

for all x, y in $\mathbb{Q}$ and $f(1) = 2002$. Justify your answers.

Alternate Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $g(x) = f(x) - x^2$ for $x \in \mathbb{Q}$. Then, for all x, $y \in \mathbb{Q}$,

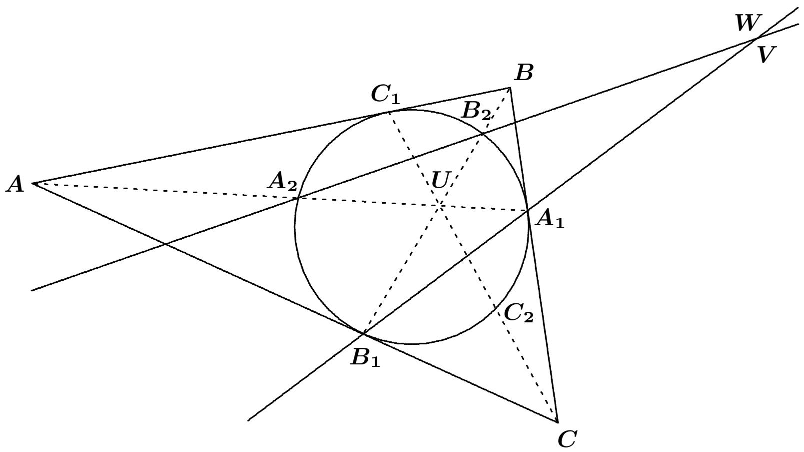
$$\begin{aligned} g(x+y) &= f(x+y) - (x+y)^2 \\ &= f(x) + f(y) + 2xy - (x+y)^2 = g(x) + g(y) . \end{aligned}$$

This is the well-known Cauchy Equation, whose solutions are $g(x) = cx$, where c is a constant. Hence, $f(x) = x^2 + 2001x$ (since $f(1) = 2002$).

Readers' solutions to some of the problems from the 38th Mongolian Mathematical Olympiad, given in [2005 : 505], were presented in the March issue of the *Corner* [2007 : 86–88]. Next we look at solutions to two problems not discussed there.

3. The incircle of triangle ABC with $AB \neq BC$ touches sides BC and AC at points A_1 and B_1, respectively. The segments AA_1 and BB_1 meet the incircle at A_2 and B_2, respectively. Prove that the lines AB, A_1B_1, and A_2B_2 are concurrent.

Solution by Michel Bataille, Rouen, France.



Let the incircle Γ touch the side AB at C_1 and let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. Since

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} \;=\; \frac{s-a}{s-c} \cdot \frac{s-c}{s-b} \cdot \frac{s-b}{s-a} \;=\; 1\,,$$

Ceva's Theorem shows that the lines AA_1, BB_1, and CC_1 are concurrent, say at U (U is the Gergonne Point of the triangle).

Let V be the pole of the line CC_1 with respect to the circle Γ. Since CA_1 and CB_1 are tangent to Γ at A_1 and B_1, respectively, the polar of C with respect to Γ is the line A_1B_1. By polar reciprocity, V is on A_1B_1. Similarly, the polar of C_1 is the line AB; hence, V is on AB. Now, let A_1B_1 and A_2B_2 meet at W. Since A_1A_2 and B_1B_2 meet at U, the polar of W with respect to Γ passes through U. But this polar also passes through C (since W is on A_1B_1). Thus, the polar of W is $CU = CC_1$ and $W = V$. Finally, V is on AB, A_1B_1, and A_2B_2 and the result follows.

6. Let A_1, B_1, and C_1 be the respective mid-points of the sides BC, AC, and AB of triangle ABC. Take a point K on the segment C_1A_1 and a point L on the segment A_1B_1 such that

$$\frac{C_1K}{KA_1} = \frac{BC+AC}{AC+AB} \qquad \text{and} \qquad \frac{A_1L}{LB_1} = \frac{AC+AB}{AB+BC}\,.$$

Let $S = BK \cap CL$. Show that $\angle C_1A_1S = \angle B_1A_1S$.

Solution by Michel Bataille, Rouen, France.

As usual, let $a = BC$, $b = CA$, and $c = AB$. Denote by $d(X, YZ)$ the distance from point X to the line YZ and by $[XYZ]$ the area of triangle XYZ. Since S is interior to $\triangle A_1B_1C_1$, the desired conclusion is successively equivalent to

$$\begin{aligned} &S \text{ is on the internal bisector of } \angle B_1A_1C_1\,, \\ d(S, A_1C_1) &= d(S, A_1B_1)\,, \\ A_1B_1 \cdot A_1C_1 \cdot d(S, A_1C_1) &= A_1C_1 \cdot A_1B_1 \cdot d(S, A_1B_1)\,, \\ c \cdot [SA_1C_1] &= b \cdot [SA_1B_1]\,. \end{aligned} \qquad (1)$$

Denote by $\overrightarrow{X}$ the vector to X from a fixed origin. From the hypotheses, we have

$$\begin{aligned} (a+2b+c)\overrightarrow{K} &= (a+b)\overrightarrow{A}_1 + (b+c)\overrightarrow{C}_1 \\ (a+b+2c)\overrightarrow{L} &= (a+c)\overrightarrow{A_1} + (b+c)\overrightarrow{B}_1 \end{aligned}$$

$$\overrightarrow{B} = \overrightarrow{A}_1 - \overrightarrow{B}_1 + \overrightarrow{C}_1 \qquad \text{and} \qquad \overrightarrow{C} = \overrightarrow{A}_1 + \overrightarrow{B}_1 - \overrightarrow{C}_1\,.$$

Thus,

$$(a+2b+c)\overrightarrow{K} - b\overrightarrow{B} \;=\; (a+b+2c)\overrightarrow{L} - c\overrightarrow{C} \;=\; a\overrightarrow{A}_1 + b\overrightarrow{B}_1 + c\overrightarrow{C}_1\,;$$

whence (since $S = BK \cap CL$),

$$(a+b+c)\overrightarrow{S} \;=\; a\overrightarrow{A}_1 + b\overrightarrow{B}_1 + c\overrightarrow{C}_1\,.$$

As a result, $a:b:c = [SB_1C_1]:[SC_1A_1]:[SA_1B_1]$, and (1) follows.

Note: Since $B_1C_1 = \frac{1}{2}a$, $C_1A_1 = \frac{1}{2}b$, and $A_1B_1 = \frac{1}{2}c$, the result just obtained even shows that S is the incentre of $\triangle A_1B_1C_1$.

Also in the March *Corner* were some readers' solutions to problems of the 19th Balkan Mathematical Olympiad, given in [2005 : 506]. For these solutions, see [2007 : 88–90]. We now present another solution.

2. The sequence a_1, a_2, ..., a_n, ... is defined by

$$a_1 \;=\; 20\,, \qquad a_2 \;=\; 30\,, \qquad a_{n+2} \;=\; 3a_{n+1} - a_n\,, \quad \text{for } n > 1.$$

Find all positive integers n for which $1 + 5a_na_{n+1}$ is a perfect square.

Solution by Michel Bataille, Rouen, France.

Since $a_3 = 70$ and $a_4 = 180$, we have $1 + 5a_3a_4 = 63001 = 251^2$. Thus, $n = 3$ is a solution. We show that there is no other solution.

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, and let $\{F_n\}$ be the Fibonacci sequence, given by $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, $(n = 0,\ 1,\ 2,\ \ldots)$. Since the solutions to the equation $x^2 - 3x + 1 = 0$ are α^2 and β^2, the classical method easily leads to

$$a_n \;=\; 10\beta^2\alpha^{2n} + 10\alpha^2\beta^{2n} \;=\; 10(\alpha^{2n-2} + \beta^{2n-2})\,,$$

where the latter equality follows from $\alpha\beta = -1$. Using this, we see that

$$1 + 5a_na_{n+1} \;=\; 501 + (50F_{2n-1})^2\,.$$

Now, if this integer is a perfect square, say K^2, we have

$$501 \;=\; (K - 50F_{2n-1})(K + 50F_{2n-1})\,.$$

Thus, either $K - 50F_{2n-1} = 3$ and $K + 50F_{2n-1} = 167$ or $K - 50F_{2n-1} = 1$ and $K + 50F_{2n-1} = 501$. The first case yields $25F_{2n-1} = 41$, which is clearly impossible. The second case gives $F_{2n-1} = 5$, which implies that $n = 3$. This completes the proof.

An eagle-eyed reader has pointed out a slight oversight in the remark given with the solution to problem 3 of the Bulgarian Mathematical Olympiad [2005 : 506–507] discussed at [2007 : 89–90].

Comment by Daniel Tsai, student, Taipei American School, Taipei, Taiwan, modified by the editor.

In the solution to problem 3 of the Bulgarian Mathematical Olympiad, Final Round, 2003, given in the March 2007 issue of ***CRUX with MAYHEM***, is the remark that $x_n = f_{2n+1}$ for all $n \geq 1$. But for $n = 1$, we have $x_1 = 1 \neq 2 = f_3$. The remark should have stated that $x_n = f_{2n-3}$ for $n \geq 2$.

Now we turn to our file of solutions from our readers to problems given in the October 2006 issue of the *Corner*. We begin with solutions to problems of the First Round of the Iranian Mathematical Olympiad given at [2006 : 372].

1. Find all permutations $(a_1, \ldots, a_n)$ of $(1, \ldots, n)$ which have the property that $i + 1$ divides $2(a_1 + \cdots + a_i)$ for every i, $1 \leq i \leq n$.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Pour $n = 1$, il n'existe évidemment qu'une seule permutation adéquate. Nous allons prouver que, pour tout $n \geq 2$, il existe exactement deux telles permutations, qui sont $(1, 2, 3, 4, \ldots, n)$ et $(2, 1, 3, 4, \ldots, n)$.

On vérifie directement que c'est le cas pour $n = 2$ et $n = 3$. De plus, il est facile de vérifier que ces deux permutations ont bien la propriété demandée pour tout $n \geq 2$.

Supposons que l'affirmation soit vraie pour $n - 1 \geq 2$. On considère une permutation $(a_1, a_2, \ldots, a_n)$ de $(1, 2, \ldots, n)$ ayant la propriété requise par l'énoncé. On va prouver que $a_n = n$. Alors l'hypothèse de récurrence assurera que $(a_1, a_2, \ldots, a_{n-1})$ est $(1, 2, 3, \ldots, n-1)$ ou $(2, 1, 3, \ldots, n-1)$.

On sait que $2(a_1+a_2+\cdots+a_n) = 2(1+2+\cdots+n) = n(n+1)$. Aussi, d'après la propriété de l'énoncé, on sait que n divise $2(a_1+a_2+\cdots+a_{n-1})$. Par suite, n divise $2a_n$.

Cas 1. Si n est impair, il vient immédiatement que n divise a_n. Et, comme $a_n \in \{1, \ldots, n\}$, c'est donc que $a_n = n$.

Cas 2. Si $n = 2k$, on doit avoir a_n divisible par k. Si $a_n \neq n$, comme $a_n < 2k$, c'est donc que $a_n = k$. Mais $n - 1 = 2k - 1$ divise

$$\begin{aligned} 2(a_1 + a_2 + \cdots + a_{n-2}) &= n(n+1) - 2a_n - 2a_{n-1} \\ &= 2k(2k+1) - 2k - 2a_{n-1} . \end{aligned}$$

Comme $2k - 1$ est impair, on en déduit que $2k - 1$ divise

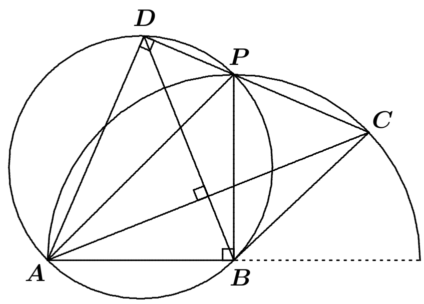
$$k(2k+1) - k - a_{n-1} \equiv k - a_{n-1} \pmod{2k-1} .$$

Puisque $a_{n-1} \in \{1, \ldots, 2k\}$, on a $-k \leq k - a_{n-1} \leq k - 1$. La seule possibilité d'avoir un multiple de $2k - 1$ est donc que $k - a_{n-1} = 0$, ou encore $a_{n-1} = k = a_n$, ce qui est impossible. Donc, on a bien $a_n = n$ et cela achève la démonstration.

4. Let A and B be two fixed points in the plane. Let $ABCD$ be a convex quadrilateral such that $AB = BC$, $AD = DC$, and $\angle ADC = 90^\circ$. Prove that there is a fixed point P such that, for every such quadrilateral $ABCD$ on the same side of the line AB, the line DC passes through P.

Solution by Michel Bataille, Rouen, France, modified by the editor.

On the same side of AB as the quadrilateral $ABCD$, draw the semi-circle (S) with centre B and radius BA, and the ray (R) originating at B and perpendicular to BA. We will show that all the lines CD pass through the point of intersection of (S) and (R).



Let $ABCD$ be an arbitrary quadrilateral satisfying the given conditions, and let P be the point of intersection of CD and (R).

We complete the proof by showing that this point P is on (S); that is, $BP = BC$. Let $\angle(\ell, \ell')$ denote the directed angle of the lines ℓ and ℓ'. Our goal will be reached if we prove the equality $\angle(PB, PC) = \angle(CP, CB)$.

We will use the fact that $\triangle ADC$ is right-angled and isosceles and that A, D, P, and B are concyclic (on the circle with diameter AP). Note also that BD is perpendicular to AC (since $BA = BC$ and $DA = DC$).

First, $\angle(PB, PC) = \angle(PB, PD) - \pi = \angle(AB, AD)$. Then,

$$\begin{aligned} \angle(CP, CB) &= \angle(CP, CA) + \angle(CA, CB) \\ &= \angle(AC, AD) + \angle(AB, AC) \\ &\qquad \text{(since } AD = DC \text{ and } AB = BC\text{)} \\ &= \angle(AB, AD)\,, \end{aligned}$$

and the result follows.

5. Let δ be a symbol such that $\delta \neq 0$ and $\delta^2 = 0$. Define

$$\begin{aligned} \mathbb{R}[\delta] &= \{a + b\delta \mid a\,,\, b \in \mathbb{R}\} \\ a + b\delta = c + d\delta &\iff a = c \text{ and } b = d\,, \\ (a + b\delta) + (c + d\delta) &= (a + c) + (b + d)\delta\,, \\ (a + b\delta) \cdot (c + d\delta) &= ac + (ad + bc)\delta\,. \end{aligned}$$

Let $P(x)$ be a polynomial with real coefficients. Show that $P(x)$ has a multiple root in $\mathbb{R}$ if and only if $P(x)$ has a non-real root in $\mathbb{R}[\delta]$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's version.

Let a, $b \in \mathbb{R}$. An easy induction shows that for all $n \in \mathbb{N}$, we have $(a+b\delta)^n = a^n + na^{n-1}b\delta$. It follows that

$$P(x+y\delta) \;=\; P(x) + P'(x)y\delta \qquad (1)$$

for all real x and y.

If $P(x)$ has a multiple root x_0 in $\mathbb{R}$, then $P(x_0) = P'(x_0) = 0$ and, from (1), we have $P(x_0+\delta) = 0$. Thus, $P(x)$ has a non-real root in $\mathbb{R}[\delta]$.

Conversely, if $P(a+b\delta) = 0$ for some real numbers a and b with $b \neq 0$, then, from (1) again, $P(a) + P'(a)b\delta = 0 = 0 + 0\delta$. Hence, $P(a) = P'(a) = 0$ and a is a multiple real root of $P(x)$.

6. Let G be a simple graph with 100 edges on 20 vertices. We can choose a pair of disjoint edges in 4050 ways. Prove that G is regular.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Soient $V_1, \ldots, V_{20}$ les sommets de G, de degrés respectifs $d_1, \ldots, d_{20}$. Il s'agit de prouver que $d_1 = \cdots = d_{20}$.

Or, on sait que la somme des degrés est le double du nombre d'arêtes, donc

$$\sum_{i=1}^{20} d_i \;=\; 200\,. \qquad (1)$$

Soit (V_i, V_j) une arête. Il y a exactement $100-(d_i+d_j-1) = 101-(d_i+d_j)$ arêtes disjointes de (V_i, V_j). Et donc autant de paires d'arêtes disjointes dont une est (V_i, V_j). En sommant sur l'ensemble des arêtes, on obtient ainsi le double du nombre de paires d'arêtes disjointes (chacune est obtenue deux fois dans le raisonnement précédent).

Il vient donc $\sum\limits_{(V_i,V_j)} [101 - (d_i + d_j)] = 2 \times 4050$, ou encore

$$\sum_{(V_i,V_j)} (d_i + d_j) \;=\; 101 \times 100 - 2 \times 4050 \;=\; 2000\,.$$

Or, dans la somme ci-dessus, chaque d_i apparaît autant de fois qu'il existe d'arêtes dont un sommet est V_i, soit donc exactement d_i fois. Par conséquent, on a

$$2000 \;=\; \sum_{(V_i,V_j)} (d_i + d_j) \;=\; \sum_{i=1}^{20} d_i^2\,.$$

Mais, d'après (1) et l'inégalité entre les moyennes arithmétiques et quadratiques (AM/QM), on a alors

$$2000 \;=\; \sum_{i=1}^{20} d_i^2 \;\geq\; \frac{1}{20}\left(\sum_{i=1}^{20} d_i\right)^2 \;=\; 2000\,.$$

On est donc dans un cas d'égalité de AM/QM, ce qui signifie que tous les d_i sont égaux, comme désiré.

Next we look at solutions from our readers to problems of the Second Round of the Iranian Mathematical Olympiad 2002 given at [2006 : 373].

1. The sequence $\{a_n\}$ is defined by $a_0 = 2$, $a_1 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$. Show that if p is a prime factor of $a_{2k} - 2$, then p is a factor of $a_{2k+1} - 1$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We first give Bataille's exposition, followed by Bornsztein's.

The sequence $\{a_n\}$ is the Lucas sequence, often associated with the Fibonacci sequence $\{f_n\}$ defined by $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$. As is well known, $a_n = \alpha^n + \beta^n$ and $f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, for $n = 0, 1, 2, \ldots$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Note that $\alpha + \beta = 1$ and $\alpha\beta = -1$. From these results, the following formulas are readily deduced:

$$a_{4n+2} - 2 = a_{2n+1}^2 , \quad (1)$$

$$a_{4n+3} - 1 = a_{2n+2}a_{2n+1} , \quad (2)$$

$$a_{4n} - 2 = 5f_{2n}^2 , \quad (3)$$

$$a_{4n+1} - 1 = 5f_{2n}f_{2n+1} . \quad (4)$$

Suppose first that k is odd, say $k = 2n + 1$ for some $n \geq 0$. If p is a prime factor of $a_{2k} - 2 = a_{4n+2} - 2$, then, from (1), p is a prime factor of a_{2n+1}^2 and hence of a_{2n+1}. Therefore, from (2), p is a prime factor of $a_{2k+1} - 1 = a_{4n+3} - 1$.

Similarly, if k is even, we deduce from (3) that a prime factor of $a_{2k} - 2$ is 5 or a prime factor of f_{2n}. In any case, as (4) shows, this prime factor divides $a_{2k+1} - 1$.

Nous donnons aussi l'approche de Bornsztein.

On pose $U_n = a_{n-1}a_{n+1} - a_n^2$. On a directement $U_1 = 5$. Et, pour tout $n \geq 1$, il vient :

$$\begin{aligned} U_{n+1} &= a_n a_{n+2} - a_{n+1}^2 = a_n a_{n+1} + a_n^2 - a_{n+1}^2 \\ &= a_{n+1}(a_n - a_{n+1}) + a_n^2 = a_{n+1}(-a_{n-1}) + a_n^2 = -U_n . \end{aligned}$$

Par conséquent, pour tout $n \geq 1$, on a $U_n = 5(-1)^{n-1}$, ou encore

$$a_{n-1}a_{n+1} - a_n^2 = 5(-1)^{n-1} . \quad (1)$$

Soit p un nombre premier qui divise $a_{2k} - 2$; donc $a_{2k} \equiv 2 \pmod{p}$. D'après (1), on a $a_{2k}^2 - 5 = a_{2k-1}a_{2k+1} = (a_{2k+1} - a_{2k})a_{2k+1}$, puis $-1 \equiv a_{2k+1}^2 - 2a_{2k+1} \pmod{p}$, ou encore $(a_{2k+1} - 1)^2 \equiv 0 \pmod{p}$. Ainsi, p divise $a_{2k+1} - 1$, comme désiré.

2. Let A be a point outside the circle Ω. The tangents from A to Ω touch Ω at B and C. A tangent L to Ω intersects AB and AC at P and Q, respectively. The line parallel to AC passing through P meets BC at R. Prove that as L varies, QR passes through a fixed point.

Comment by Michel Bataille, Rouen, France.

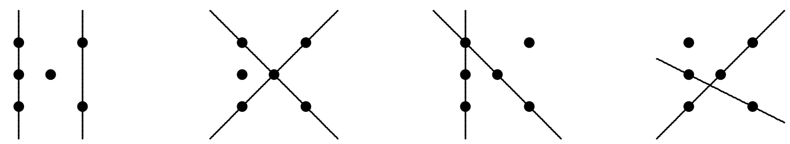
This problem is not new. It is problem 2639 ([2001 : 268; 2002 : 272]). Two different solutions were given in the May 2002 issue.

4. Find the smallest positive integer n for which the following condition holds: For every finite set of points in the plane, if, for every n points in this set, there exist two lines covering all n points, then there exist two lines covering all points in the set.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Le plus petit entier n cherché est $n = 6$.

Soit $\mathcal{E}_5$ l'ensemble formé des points $A(0,0)$, $B(2,0)$, $C(2,2)$, $D(0,2)$, $M(0,1)$ et $\Omega(1,1)$. Les schémas suivant montrent que toute partie à 5 éléments de $\mathcal{E}_5$ peut être recouverte par deux droites.



Par contre, $\mathcal{E}_5$ ne peut être recouvert par deux droites : en effet, comme $\mathcal{E}_5$ ne contient pas quatre points alignés, tout recouvrement éventuel de $\mathcal{E}_5$ par deux droites se ferait par deux droites contenant chacune trois points. Le seul groupe de trois points alignés contenant M est A, M, D. Ainsi, c'est nécessairement l'un des deux groupes qui définit une des deux droites. L'autre doit alors passer par les trois points restant, mais ceux-ci ne sont pas alignés. Cela prouve que $n = 5$ n'a pas la propriété désirée.

On prouve maintenant $n = 6$ possède la propriété de l'énoncé. Cela étant, il convient d'intérpréter cette condition en :

> For every finite set of at least n points in the plane, if, for every n points in this set, there exist two lines covering all n points, then there exist two lines covering all points in the set.

Sinon, il n'existe aucun entier vérifiant la condition demandée (prendre l'ensemble des sommets d'un pentagone régulier. Pour $n \geq 6$, il n'existe aucun ensemble de n points dans cet ensemble donc les prémisses sont trivialement vérifiées, mais l'ensemble ne peut être recouvert par deux droites).

Soit $\mathcal{E}$ un ensemble d'au moins 6 points tel que tout sous-ensemble de 6 points de $\mathcal{E}$ puisse être recouvert par deux droites.

Soit $\mathcal{F} \subset \mathcal{E}$, avec $|\mathcal{F}| = 6$. Comme $\mathcal{F}$ peut être recouvert par deux droites, le principe des tiroirs assure qu'au moins trois des points de $\mathcal{F}$, et donc de $\mathcal{E}$, sont alignés. Disons que A, B, C sont trois points, deux à deux distincts, alignés et appartenant à $\mathcal{E}$. On note Δ la droite (AB).

Si $\mathcal{E} - \Delta$ ne contient pas plus de deux points, alors $\mathcal{E}$ peut clairement être recouvert par deux droites. Si $\mathcal{E} - \Delta$ contient au moins trois éléments : Soient X et Y deux points distincts dans $\mathcal{E} - \Delta$. On note Δ' le droite (XY).

Pour tout point $M \in \mathcal{E} - \Delta$, autre que X et Y, l'ensemble formé des points A, B, C, X, Y, M peut être recouvert par deux droites. Si aucune de ces deux droites n'est Δ, celle qui recouvre A ne recouvre ni B ni C. Donc l'autre doit recouvrir à la fois B et C, mais alors c'est Δ, une contradiction.

Ainsi, l'une des deux droites est Δ. l'autre doit nécessairement recouvrir les points X, Y et M, ce qui prouve que $M \in \Delta'$.

Par conséquent, tout point de $\mathcal{E}$ appartient à Δ ou à Δ', et donc $\mathcal{E}$ peut être recouvert par deux droites.

6. Let a, b, and c be positive real numbers such that $a^2+b^2+c^2+abc = 4$. Prove that $a+b+c \leq 3$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Vedula N. Murty, Dover, PA, USA. We first give the solution of Díaz-Barrero, using a classic change of variables and geometry.

Setting $a = 2\cos A$, $b = 2\cos B$, and $c = 2\cos C$, with $A+B+C = \pi$, we have

$$\begin{aligned} & a^2+b^2+c^2+abc \\ &= 4\cos^2 A + 4\cos^2 B + 4\cos^2(A+B) - 8\cos A\cos B\cos(A+B) \\ &= 4\cos^2 A + 4\cos^2 B - 4\cos^2 A\cos^2 B + 4\sin^2 A\sin^2 B \\ &= 4\sin^2 B(\cos^2 A + \sin^2 A) + 4\cos^2 B \;=\; 4\,. \end{aligned}$$

Taking into account Euler's Inequality, $R \geq 2r$, and the well-known identity $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$, we get

$$a+b+c \;=\; 2(\cos A + \cos B + \cos C) \;=\; 2\left(1+\frac{r}{R}\right) \;\leq\; 3\,.$$

Note that equality holds when $a = b = c = 1$.

Next we give the solution of Murty.

Without loss of generality, we assume that $0 < a \leq b \leq c$. From $a^2+b^2+c^2+abc = 4$ we deduce that $0 < a \leq 1$, $0 < b < 2$, and $1 \leq c < 2$. Now $c^2 + c(ab) + a^2 + b^2 - 4 = 0$ is a quadratic equation in c and the positive root is

$$c \;=\; \tfrac{1}{2}\left(-ab + \sqrt{(4-a^2)(4-b^2)}\right)\,.$$

Hence, $a+b+c \leq 3$ if and only if

$$\sqrt{(4-a^2)(4-b^2)} \;\leq\; 6 - 2a - 2b + ab\,. \tag{1}$$

From the AM–GM Inequality, we have

$$\sqrt{(4-a^2)(4-b^2)} \;\leq\; \tfrac{1}{2}\left(8 - (a^2+b^2)\right)\,.$$

We now prove that

$$\tfrac{1}{2}(8-(a^2+b^2)) \ \le\ 6-2a-2b+ab\,. \qquad (2)$$

This inequality is equivalent to $(a+b)^2-4(a+b)+4\ge 0$, which factors as $(a+b-2)^2\ge 0$. Thus (2) is true. Then (1) is true. Equality is attained when $a=b=c=1$.

Finally, we look at solutions from our readers to problems of the Third Round of the Iranian Mathematical Olympiad 2002 given in [2006 : 373–374].

1. Find all real polynomials $P(x)$ such that $P(a)\in\mathbb{Z}$ implies that $a\in\mathbb{Z}$.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Les solutions sont les polynômes constants et ceux de la forme $P(x)=(x+b)/c$, où b et c sont deux entiers avec $c\neq 0$.

Il est facile de vérifier que les polynômes ci-dessus conviennent effectivement.

Soit P un polynôme non constant ayant la propriété de l'énoncé. Quitte à changer P en $-P$, on peut supposer que

$$\lim_{x\to+\infty} P(x) \ =\ +\infty\,. \qquad (1)$$

Cas 1. On suppose que P est de degré au moins égal à 2.

D'après (1), on a alors

$$\lim_{x\to+\infty} P'(x) \ =\ +\infty\,. \qquad (2)$$

D'après (1) et (2), et puisqu'il s'agit de polynômes, il existe donc $A>0$ tel que P et P' soient strictement croissants sur $[A,+\infty)$. Toujours d'après (2), on peut également supposer que

$$P'(x)>2 \ \text{ pour } \ x\ge A. \qquad (3)$$

Soit q un entier tel que $q\ge P(A)$. D'après la propriété de l'énoncé et le théorème des valeurs intermédiaries, pour tout entier $n\ge 0$, il existe un entier $x_n\ge A$ tel que

$$P(x_n) \ =\ q+n\,. \qquad (4)$$

Puisque P est strictement croissant sur $[A,+\infty)$, la suite $\{x_n\}$ est donc strictement croissante. En particulier, s'agissant d'entiers, on a

$$x_{n+1}-x_n \ \ge\ 1 \quad \text{pour tout } n\ge 0. \qquad (5)$$

D'après le théorème des accroissements finis, pour tout entier $n\ge 0$, il existe $y_n\in[x_n,x_{n+1}]$ tel que $P(x_{n+1})-P(x_n)=(x_{n+1}-x_n)P'(y_n)$.

D'après (3) et (5), on a donc $P(x_{n+1}) - P(x_n) > 2$. Mais, d'après (4), on a $P(x_{n+1}) - P(x_n) = 1$, une contradiction. Il n'existe donc aucun polynôme de degré supérieur ou égal à 2 possédant la propriété désirée.

Cas 2. On suppose que $P(x) = \alpha x + \beta$, où α et β sont des réels et $\alpha > 0$. Alors, d'après la propriété de l'énoncé, ils existent deux entiers x_0 et x_1 tels que $P(x_0) = 0$ et $P(x_1) = 1$. Donc $\alpha x_0 + \beta = 0$ et $\alpha x_1 + \beta = 1$, d'où $\alpha(x_1 - x_0) = 1$. On a $x_1 - x_0 > 0$, puisque $\alpha > 0$. Soit $c = x_1 - x_0$. Alors $\alpha = 1/c$ et $\beta = -x_0/c$, ce qui prouve que P est bien de la forme annoncé et achève la démonstration.

3. In a triangle ABC, define C_a to be the circle tangent to AB, to AC, and to the incircle of the triangle ABC, and let r_a be the radius of C_a. Define r_b and r_c in the same way. Prove that $r_a + r_b + r_c \geq 4r$, where r is the inradius of the triangle ABC.

Solution by Michel Bataille, Rouen, France.

The given inequality is false (for example $r_a = r_b = r_c = \frac{r}{3}$ in an equilateral triangle). We will prove instead that

$$r_a + r_b + r_c \geq r . \qquad (1)$$

The circle C_a is the image of the incircle in the homothety with centre A and scale factor r_a/r. Hence, if I is the incentre and I_a is the centre of C_a, we have $\overrightarrow{AI_a} = \frac{r_a}{r}\overrightarrow{AI}$, which can be rewritten as

$$\overrightarrow{II_a} = \left(\frac{r_a}{r} - 1\right)\overrightarrow{AI} .$$

Since $r_a < r$ and $II_a = r + r_a$, it follows that

$$\frac{r_a}{r} = \frac{AI - r}{AI + r} = \frac{2AI}{AI + r} - 1 = \frac{2}{1 + (r/AI)} - 1 = \frac{2}{1 + \sin(A/2)} - 1 .$$

Similar results hold for r_b and r_c. Thus, we see that (1) is equivalent to

$$\frac{1}{1 + \sin(A/2)} + \frac{1}{1 + \sin(B/2)} + \frac{1}{1 + \sin(C/2)} \geq 2 . \qquad (2)$$

The function $f(x) = 1/(1+\sin x)$ is strictly convex on $\left(0, \frac{\pi}{2}\right)$ (its second derivative is $f''(x) = (1 + \sin x)^{-3}(1 + \sin x + \cos^2 x)$, which is positive); hence,

$$f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \geq 3f\left(\frac{A + B + C}{3}\right) = 3f\left(\frac{\pi}{6}\right) = 2 .$$

Therefore, (2) holds. Equality is attained in (2) if and only if $A = B = C$; that is, if and only if $\triangle ABC$ is equilateral.

That completes the material for this number of the *Corner*. Please send your nice generalizations and solutions as well as Olympiad Contests.